

Cash Sub-additive Risk Measures and Interest Rate Ambiguity*

Nicole El Karoui^a

Claudia Ravanelli^b

^aCMAP, Ecole Polytechnique, France

^bSwiss Banking Institute, University of Zurich, Switzerland

December 6, 2008

Abstract

A new class of risk measures called cash sub-additive risk measures is introduced to assess the risk of future financial, nonfinancial and insurance positions. The debated cash additive axiom is relaxed into the cash sub-additive axiom to preserve the original difference between the numéraire of the current reserve amounts and future positions. Consequently, cash sub-additive risk measures can model stochastic and/or ambiguous interest rates or defaultable contingent claims. Practical examples are presented and in such contexts cash additive risk measures cannot be used. Several representations of the cash sub-additive risk measures are provided. The new risk measures are characterized by penalty functions defined on a set of sub-linear probability measures and can be represented using penalty functions associated with cash additive risk measures defined on some extended spaces. The issue of the optimal risk transfer is studied in the new framework using inf-convolution techniques. Examples of dynamic cash sub-additive risk measures are provided via BSDEs where the generator can locally depend on the level of the cash sub-additive risk measure.

Keywords: Risk measures, Fenchel-Legendre transform, model uncertainty, inf-convolution, backward stochastic differential equations.

JEL Classifications: D81, G13.

*Address correspondence to Claudia Ravanelli, Swiss Banking Institute, University of Zurich, Plattenstrasse 14, CH-8031 Zurich, Switzerland. E-mail: ravanelli@isb.uzh.ch. E-mail address for Nicole El Karoui: elkaroui@cmappx.polytechnique.fr. For valuable comments we thank seminar participants at SAMSI 2006 workshop, Evry Workshop 2006, Princeton 2006 conference on Risk Measures and Robust Control in Finance and Cass Business School. We are grateful to the anonymous referees for many useful and fruitful suggestions which improved the paper. Ravanelli's research was supported by the University Research Priority Program "Finance and Financial Markets" (University of Zurich) and by the NCCR-FinRisk (Swiss National Science Foundation).

1 Introduction

The assessment of financial and nonfinancial risks plays a key role for economic agents when pricing assets or managing their wealths. Consequently, over the last decade several measures of risk have been proposed to assess the riskiness of financial and nonfinancial positions and compute cash reserve amounts for hedging purposes. The axiomatic based monetary risk measures have been largely investigated because most axioms embed desirable economic properties. Coherent risk measures have been introduced by Artzner, Delbaen, Eber, and Heath (1997), Artzner, Delbaen, Eber, and Heath (1999), and further developed by Delbaen (2001), Delbaen (2002); sublinear risk measures by Frittelli (2000); convex risk measures by Föllmer and Schied (2002a), Föllmer and Schied (2002b) and Frittelli and Rosazza Gianin (2002). Examples of convex risk measures related to pricing and hedging in incomplete markets are provided by, for instance, El Karoui and Quenez (1996), Carr, Geman, and Madan (2001), Frittelli and Rosazza Gianin (2004) and Staum (2004). However, while the convexity and the monotonicity axioms have been largely accepted by academics and practitioners, the cash additive axiom has been criticized from an economic viewpoint. A basic reason is that while regulators and financial institutions determine and collect today the reserve amounts to cover future risky positions, the cash additivity requires that risky positions and reserve amounts are expressed in the same numéraire. This is a stringent requirement that limits the applicability of cash additive risk measures. Implicitly it means that risky positions are discounted before applying the risk measure assuming that the discounting process does not involve any additional risk. Unfortunately, when the interest rates are stochastic this procedure does not disentangle the risk of the financial position per se and the risk associated to the discounting process¹. Furthermore, payoff functions on risky assets are a priori and contractually determined by economic agents considering different scenarios for the underlying asset. While this procedure is theoretically framed into cash additive risk measures, the cash additive axiom does not allow to account for ambiguous discount factor. For a correct assessment of the current reserve amount it is equally important to allow for ambiguity on the underlying asset and on the discount factor. This assessment is achieved by relaxing the cash additive axiom and searching for risk measures that preserve the different numéraires of the current reserve amounts and the future risky positions.

¹Disentangling the different risks is crucial when implementing hedging strategies as different risks are hedged on different markets.

The main contribution of this paper is to propose a new class of risk measures called cash sub-additive risk measures that are directly defined on the future risky positions and provide the reserve amounts in terms of the current numéraire. To reconcile the two different numéraires cash sub-additive risk measures relax the cash additive axiom into the cash sub-additive axiom. This is the minimal requirement to account for the time value of money. Remarkably, the cash sub-additive axiom is enough to characterize measures of risk that can be applied also when the cash additive risk measures cannot—as for instance under ambiguous interest rates or defaultable cash flows. Cash sub-additive risk measures turn out to be suitable not only for assessing financial risks but also insurance and other kind of risks. For example, the put option premium investigated by Jarrow (2002) as a measure of the firm insolvency risk defines a cash sub-additive risk measure. Moreover, similarly to the cash additive risk measures, the cash sub-additive risk measures can be represented using penalty functions. In particular, we show that cash sub-additive risk measures are characterized by minimal penalty functions which only depend on finitely additive set functions μ such that $0 \leq \mu(\Omega) \leq 1$, that we call finitely additive sub-probability measures.

The other contributions of this paper are the following. In the framework of cash additive risk measures when the zero-coupon bond is available for the relevant time horizon, we provide the conditions under which discounting the forward risk measure to obtain current reserve amounts defines risk measures additive with respect the current numéraire and vice versa (Section 2.4).

In Section 3 we introduce the cash sub-additive risk measures that we denote by \mathcal{R} . We provide several examples of these new risk measures that generalize the put option premium and naturally arise when accounting for ambiguous discount factor or insurance risks. These risk measures are obtained composing cash additive risk measures and a specific class of random convex functions. A representation result showing the impact of the ambiguous discount factor/numéraire is given.

In Section 4 we study the dual representation of cash sub-additive risk measures. Instead of using convex analysis tools, we extend cash sub-additive risk measures to an enlarged space of risky positions where they become cash additive. This approach provides a rich financial interpretation of both cash additive and cash sub-additive risk measures and allows to derive properties of \mathcal{R} using the classical theory on cash additive risk measures. Using the duality result, a characterization of \mathcal{R} in terms of deterministic discount factors is easily obtained where any cash sub-additive risk measure can be represented as the worst case scenario of a family of

discounted forward risk measures.

In Section 5 two other links between cash sub-additive and cash additive risk measures are presented where more involved techniques are required. The first link indicates a possible way to recover a representation of a general cash sub-additive risk measure where the ambiguous numéraire is explicitly modeled as a random variable on the original space of definition of \mathcal{R} . The second link shows that cash sub-additive risk measures generated via convex functions are compositions of an unconditional and a conditional cash additive risk measures.

In Section 6 using cash sub-additive risk measures we study the problem of designing the optimal transaction between two economic agents in a general framework allowing for ambiguous discount factors. In particular we show that the risk transfer problem can be reduced to an inf-convolution of cash sub-additive risk measures which is again a cash sub-additive risk measure.

Finally, in Section 7 we provide a dynamic example of cash sub-additive risk measures which are solutions of backward stochastic differential equations (BSDEs). In contrast to the cash additive risk measures generated via BSDEs, the generator of dynamic cash sub-additive risk measures, besides being a function of the martingale part, can also depend on the level of the cash sub-additive risk measure, generating recursive risk measures. Section 8 concludes.

2 Cash additive risk measures

In this section we recall some key properties of cash additive risk measures and we discuss the cash additive axiom. The following definitions are consistent with the definitions of monetary risk measure in Föllmer and Schied (2002b).

2.1 Definitions and properties of cash additive risk measures

Let (Ω, \mathcal{A}) be a measurable space. The risky positions at the relevant time horizon belong to the linear space of bounded functions including constant functions denoted by \mathcal{X} .

Definition 2.1 *A cash additive risk measure is a functional $\rho : \mathcal{X} \rightarrow \mathbb{R}$ cash additive, convex and monotone decreasing, i.e.,*

$$a) \text{ Convexity: } \forall \lambda \in [0, 1], \quad \rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y);$$

$$b) \text{ Monotonicity: } X \leq Y \Rightarrow \rho(X) \geq \rho(Y);$$

c) Cash additivity (or cash invariance): $\forall m \in \mathbb{R}, \quad \rho(X + m) = \rho(X) - m$.

A cash additive risk measure is *coherent* when

d) Positive homogeneity: $\forall \lambda \in \mathbb{R}^+, \quad \rho(\lambda X) = \lambda \rho(X)$.

e) ρ is normalized when $\rho(0) = 0$.

f) ρ is continuous from below (from above) when

$$X_n \nearrow X \Rightarrow \rho(X_n) \searrow \rho(X), \quad (X_n \searrow X \Rightarrow \rho(X_n) \nearrow \rho(X)).$$

The convexity axiom translates the natural important fact that diversification should not increase risk. In particular, convex combinations of “admissible” risks should be “admissible”.

To shorten the representation of convex combinations of elements we use the following notation.

We denote the *barycenter* (or *convex combination*) of the set $x_I := \{x_{(1)}, x_{(2)}, \dots, x_{(I)}\}$, $I \in \mathbb{N}$,

$$(2.1) \quad \text{Bar}[x_I] := \text{Bar}^{\lambda_I}[x_I] := \sum_{i=1}^I \lambda_i x_{(i)} \text{ where } \lambda_i \in [0, 1], i = 1, \dots, I, \text{ and } \sum_{i=1}^I \lambda_i = 1.$$

In particular, f is a convex function if and only if $f(\text{Bar}[x_I]) \leq \text{Bar}[f(x)_I]$. The same definition holds for a set X_I of random variables.

2.2 Dual representation of cash additive risk measures

A key property of cash additive risk measures is the dual representation in terms of normalized finitely additive set functions and minimal penalty functional (Föllmer and Schied (2002b, Theorem 4.12)). The dual point of view emphasizes the interpretation in terms of a worst case scenario related to the agent’s (or regulator’s) beliefs: the agent does not know the true “probability” measure and uses distorted beliefs from a subjective set of normalized additive set functions. Under the additional assumption that risk measures are continuous from below, the dual representation is in term of σ -additive probability measures (Föllmer and Schied (2002b, Proposition 4.17)).

Theorem 2.2 (a) Let $\mathcal{M}_{1,f}(\mathcal{A})$ be the set of all finitely additive set functions Q on (Ω, \mathcal{A}) normalized to one, $Q(\Omega) = 1$, and α the minimal penalty functional taking values in $\mathbb{R} \cup \{+\infty\}$:

$$(2.2) \quad \forall Q \in \mathcal{M}_{1,f}(\mathcal{A}), \quad \alpha(Q) = \sup_{X \in \mathcal{X}} \{\mathbb{E}_Q[-X] - \rho(X)\}, \quad (\geq -\rho(0))$$

$$(2.3) \quad \text{Dom}(\alpha) = \{Q \in \mathcal{M}_{1,f}(\mathcal{A}) \mid \alpha(Q) < +\infty\}.$$

The Fenchel duality relation holds:

$$(2.4) \quad \forall X \in \mathcal{X}, \quad \rho(X) = \sup_{Q \in \mathcal{M}_{1,f}(\mathcal{A})} \{\mathbb{E}_Q[-X] - \alpha(Q)\}.$$

Moreover, for any $X \in \mathcal{X}$ there exists a $Q_X \in \mathcal{M}_{1,f}(\mathcal{A})$, such that $\rho(X) = \mathbb{E}_{Q_X}[-X] - \alpha(Q_X) = \max_{Q \in \mathcal{M}_{1,f}(\mathcal{A})} \{\mathbb{E}_Q[-X] - \alpha(Q)\}$.

(b) Let $\mathcal{M}_1(\mathcal{A})$ denote the set of all probability measures \mathbb{Q} on (Ω, \mathcal{A}) . Let ρ be a monetary risk measure continuous from below (Fatou property) and suppose that β is any penalty function on $\mathcal{M}_{1,f}(\mathcal{A})$ representing ρ . Then β is concentrated on the class $\mathcal{M}_1(\mathcal{A})$ of probability measures, i.e., $\beta(Q) < \infty$ only if Q is σ -additive.

See Krätschmer (2005) for necessary conditions to obtain representation results in terms of probability measures.

The following lemma shows that a cash additive risk measure is linear with respect to the linear subspace generated by a position Y if and only if any Q in the domain of the penalty functional satisfies the calibration constraint: $Q(-Y) = \rho(Y)$. This lemma will be used to derive the results in Section 2.4.

Lemma 2.3 *Let ρ be a normalized cash additive risk measure on \mathcal{X} and \mathcal{W} a linear subspace of \mathcal{X} containing the constants. The risk measure ρ is a linear on \mathcal{W} , if and only if $\rho(W) = \mathbb{E}_Q[-W]$ for any $Q \in \text{Dom}(\alpha)$. This implies that the risk measure is invariant with respect to \mathcal{W} , that is $\forall X \in \mathcal{X}, \forall W \in \mathcal{W}, \rho(X + W) = \rho(X) + \rho(W)$.*

Proof. The dual representation and the linearity of ρ with respect to \mathcal{W} imply that for any $Q \in \text{Dom}(\alpha)$, $\lambda \in \mathbb{R}$, $\lambda\rho(W) = \rho(\lambda W) \geq \mathbb{E}_Q[\lambda(-W)] - \alpha(Q)$, where α is the minimal penalty of ρ . Then $\alpha(Q) \geq -\lambda(\rho(W) + \mathbb{E}_Q[W])$. As the last inequality holds for any $\lambda \in \mathbb{R}$, $\rho(W) = -\mathbb{E}_Q[W]$, $\forall Q \in \text{Dom}(\alpha)$. The vice versa is evident.

If the calibration constraint holds, then $\rho(X + W) = \sup_{Q \in \text{Dom}(\alpha)} \{\mathbb{E}_Q[-X - W] - \alpha(Q)\} = \sup_{Q \in \text{Dom}(\alpha)} \{\rho(W) + \mathbb{E}_Q[-X] - \alpha(Q)\} = \rho(W) + \rho(X)$, for any $X \in \mathcal{X}$, $W \in \mathcal{W}$. \square

2.3 Cash additivity and discounting

The cash additive axiom is motivated by the interpretation of $\rho(X)$ as capital requirement². Intuitively, $\rho(X)$ is the amount of cash which has to be added to the risky position X in order to make it acceptable (i.e., with non positive measure of risk) by a supervising agency

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.$$

Hence the cash additive property requires that the risky position and the risk measure are expressed in the same numéraire. Then either cash additive risk measures are defined on the discounted value of the future positions (see, for instance, Delbaen (2001) and Föllmer and Schied (2002b)) or cash additive risk measures are defined directly on the future positions and give the forward reserve amount to add to the future position at the future date (see, for instance, Rouge and El Karoui (2000)). In the next section, assuming that all the agents use the same discount factor for the maturity of interest and there exists a zero coupon bond for that maturity, we provide a link between cash additive risk measures on the discounted positions and forward cash additive risk measures.

In the sequel, (Ω, \mathcal{F}_T) is a measurable space and the risky position belongs to \mathcal{X} , the linear space of real-valued bounded random variables on (Ω, \mathcal{F}_T) including constants. The riskiness of $X_T \in \mathcal{X}$ is assessed at time $t = 0$ and 1_T denotes one unit of cash available at date T . D_T is the stochastic (non-ambiguous) discount factor for the maturity T used by all agents in the market. When available on the market, $B_{0,T} > 0$ denotes the price at time $t = 0$ of a zero coupon bond that pays 1 unit of cash at time $t = T$.

2.4 Forward and spot risk measures under stochastic discount factor

The following definitions of risk measures highlight with respect to which numéraire the risk measures are cash additive.

Definition 2.4 *a)* Let D_T be the \mathcal{F}_T -measurable, $0 \leq D_T \leq 1$, discount factor used by all agents in the market. A *spot risk measure*, ρ_0 , is a cash additive risk measure defined on the discounted position $D_T X_T$, $X_T \in \mathcal{X}$. The spot cash additive property is with respect to the

²See Frittelli and Scandolo (2006) for an extensive study of the axiom of cash additivity and the related concept of capital requirement.

cash available at time $t = 0$, $\forall X_T \in \mathcal{X}$,

$$(2.5) \quad \forall m \in \mathbb{R}, \quad \rho_0(D_T X_T + m) = \rho_0(D_T X_T) + \rho_0(m) \quad \text{and} \quad \rho_0(m) = m\rho_0(1) = -m.$$

b) A *forward risk measure*, ρ_T , is a cash additive risk measure defined on the future position $X_T \in \mathcal{X}$. The forward cash additive property is with respect to cash available at time T , $\forall X_T \in \mathcal{X}$,

$$(2.6) \quad \forall m \in \mathbb{R}, \quad \rho_T(X_T + m1_T) = \rho_T(X_T) + \rho_T(m1_T) \quad \text{and} \quad \rho_T(m1_T) = m\rho_T(1_T) = -m1_T.$$

The spot risk measure ρ_0 is the monetary risk measure defined in Föllmer and Schied (2002b). It represents the cash amount at $t = 0$ to add to the discounted position $D_T X_T$ to make it acceptable. The spot risk measure does not disentangle the discounting risk from the risk of the financial position per sé. Furthermore, to meaningful consider the discounted future position the discount factor cannot be ambiguous. The forward risk measure ρ_T gives the forward cash amount (evaluated at $t = 0$) to add at $t = T$ to the position to make it acceptable. When the zero coupon bond $B_{0,T}$ is available, the forward reserve $\rho_T(X_T)$ can be easily discounted at $t = 0$. The following proposition shows that this procedure defines a spot risk measure when ρ_T satisfies a calibration constraint on D_T and $B_{0,T}$. Similarly, the spot risk measure ρ_0 capitalized by $B_{0,T}^{-1}$ defines a forward risk measure if ρ_0 satisfies a similar calibration constraint on D_T and $B_{0,T}$. The penalty function of ρ_0 is equal to the penalty function of ρ_T discounted by $B_{0,T}$ and the corresponding additive set functions satisfy the usual spot-forward change of measure.

Proposition 2.5 1) Let ρ_0 be a normalized spot risk measure with minimal penalty function α_0 . The functional

$$(2.7) \quad q_T(X_T) := B_{0,T}^{-1} \rho_0(D_T X_T), \quad X_T \in \mathcal{X},$$

is convex and monotone decreasing with respect to X_T , and forward cash additive if and only if ρ_0 satisfies the calibration constraint, $\forall \lambda \in \mathbb{R}, \quad \rho_0(\lambda D_T) = -\lambda B_{0,T}$. In this case, any $Q_0 \in \text{Dom}(\alpha_0)$ is such that $\mathbb{E}_{Q_0}[D_T] = -B_{0,T}$. Moreover, if D_T is bounded away from 0, q_T satisfies the calibration constraint, $\forall \lambda \in \mathbb{R}, \quad q_T(\lambda D_T^{-1}) = B_{0,T}^{-1} \rho_0(\lambda) = -\lambda B_{0,T}^{-1} = \lambda q_T(D_T^{-1})$, and the

minimal penalty functional of q_T , α_T , is given by

$$(2.8) \quad \alpha_T(Q_T) = B_{0,T}^{-1} \alpha_0(Q_0), \forall Q_T : dQ_0 = \frac{B_{0,T}}{D_T} dQ_T \in \text{Dom}(\alpha_0), \text{ and } \alpha_T = \infty \text{ otherwise.}$$

2) Let the discount factor D_T be bounded away from 0. Given a normalized forward risk measure ρ_T with penalty function α_T , the functional

$$(2.9) \quad q_0(Y) := B_{0,T} \rho_T(Y D_T^{-1}), \quad Y \in \mathcal{X}$$

is convex and monotone decreasing with respect to Y and satisfies, $\forall \lambda \in \mathbb{R}$, $q_0(\lambda D_T) = B_{0,T} \rho_T(\lambda) = -\lambda B_{0,T} = \lambda q_0(D_T)$. Moreover, q_0 is a spot risk measure if and only if ρ_T satisfies: $\rho_T(\lambda D_T^{-1}) = -\lambda B_{0,T}^{-1} = \lambda \rho_T(D_T^{-1})$, $\forall \lambda \in \mathbb{R}$.

Proof. 1) If ρ_0 satisfies $\rho_0(\lambda D_T) = -\lambda B_{0,T}$, $\forall \lambda \in \mathbb{R}$, the forward cash additivity of q_T follows directly from Lemma 2.3. Conversely, let q_T be cash additive. This is equivalent to require that ρ_0 satisfies

$$(2.10) \quad \forall X_T \in \mathcal{X}, \forall \lambda \in \mathbb{R}, \quad \rho_0(D_T X_T + \lambda 1_T D_T) = \rho_0(D_T X_T) - \lambda B_{0,T}.$$

Setting $X_T = 0$ in (2.10) gives the result. To prove (2.8) we observe that if q_T in (2.7) is a spot risk measure with minimal penalty function α_T , the definition of the minimal penalty function and Lemma 2.3 give

$$\alpha_T(Q_T) = \sup_{X_T} \{ \mathbb{E}_{Q_T}[-X_T] - q_T(X_T) \} = \sup_{X_T} \{ B_{0,T}^{-1} \mathbb{E}_{Q_0}[-D_T X_T] - B_{0,T}^{-1} \rho_0(D_T X_T) \}.$$

Since D_T is bounded away from 0, the one to one correspondence between bounded variables and discounted bounded variables implies $\alpha_T(Q_T) = B_{0,T}^{-1} \alpha_0(Q_0)$, where $dQ_0 = D_T^{-1} B_{0,T} dQ_T$. It follows that Q_T is in the domain of α_T if and only if Q_0 is a set function in the domain of α_0 and satisfies the calibration constraint in (2.8). Conversely, a risk measure with minimal penalty functional α_T satisfying (2.8) is of the form $\rho_T(X_T) = B_{0,T}^{-1} \rho_0(D_T X_T)$.

2) Similar arguments can be used to prove the vice versa. \square

Unfortunately, the procedure of computing current reserve amounts discounting forward risk measures (given by q_0 in equation (2.9)) is feasible only when the zero coupon bonds for the relevant maturities are available on the market. In this case the functional q_0 in equation (2.9) is an example of the general capital requirement defined in Frittelli and Scandolo (2006).

Next section contains the major contribution of this paper which is the introduction of a new class of risk measures called cash sub-additive risk measures. These risk measures provide reserve amounts which account for the ambiguity on the discount factor. This result is achieved by simply relaxing the cash additive axiom into the cash sub-additive axiom and preserving the original difference in the numéraires of reserves and future positions. This will be illustrated by several examples in the finance and insurance frameworks.

3 Cash sub-additive risk measures

The following observation provides the intuition for introducing cash sub-additive risk measures. Given the (stochastic) discount factor $0 \leq D_T \leq 1$ and a spot risk measure ρ_0 in equation (2.5), the convex, non-increasing functional defined on \mathcal{X} denoted by $\mathcal{R}(X_T) = \rho_0(D_T X_T)$ is cash *sub-additive*, that is, it satisfies the following inequality: $\forall m \geq 0$,

$$\mathcal{R}(X_T + m1_T) = \rho_0(D_T X_T + D_T m) \geq \rho_0(D_T X_T + m) = \rho_0(D_T X_T) - m = \mathcal{R}(X_T) - m.$$

This inequality is a simple consequence of the time value of the money, i.e. $D_T m \leq m$. The functional \mathcal{R} is expressed in terms of the current numéraire but directly defined on the future position expressed in terms of the future numéraire. The function $m \in \mathbb{R} \mapsto \mathcal{R}(X_T 1_T + m 1_T) + m$ is non-decreasing, that is \mathcal{R} is cash sub-additive. This observation highlights the cash sub-additive axiom as the minimal condition (imposed by the time value of the money) that has to be satisfied by risk measures which preserve the two different numéraires of current reserve amounts and future risky positions. Remarkably, replacing the cash additive axiom with the cash sub-additive axiom is sufficient to characterize risk measures that can be used also when cash additive risk measures cannot. For instance under stochastic and/or ambiguous interest rates or assessing the risk of defaultable contingent claims. In the sequel we formally define the cash sub-additive risk measures denoted by \mathcal{R} . Then we provide several examples showing the different applications of these new risk measures. The previous considerations and the following examples motivate the study of cash sub-additive risk measures.

3.1 Definition of cash sub-additive risk measures

Definition 3.1 A *cash sub-additive risk measure* \mathcal{R} is a functional $\mathcal{R} : \mathcal{X} \rightarrow \mathbb{R}$, convex and non increasing satisfying the cash sub-additive axiom:

$$\forall m \in \mathbb{R}, \mathcal{R}(X_T + m1_T) + m \text{ is non decreasing in } m.$$

A cash sub-additive risk measure \mathcal{R} is *coherent* when $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$, $\forall \lambda \geq 0$. A cash sub-additive risk measure \mathcal{R} is *normalized* when $\mathcal{R}(0) = 0$.

The cash sub-additive axiom can also be expressed:

$$\forall m \in \mathbb{R}, \mathcal{R}(X_T + |m|1_T) \geq \mathcal{R}(X_T) - |m| \quad \text{and} \quad \mathcal{R}(X_T - |m|1_T) \leq \mathcal{R}(X_T) + |m|.$$

Cash sub-additive risk measures naturally account for the time value of money. When m dollars are added to the future position X_T , $X_T + m1_T$, the capital requirement at time $t = 0$ is reduced by less than m dollars, that is $\mathcal{R}(X_T1_T + m1_T) \geq \mathcal{R}(X_T1_T) - m$.

3.2 Examples of cash sub-additive risk measures

This section provides several examples of cash sub-additive risk measures. All these risk measures can be obtained composing cash additive risk measures and convex real (random) functions. The first example arises naturally considering an ambiguous discount factor.

3.2.1 Cash sub-additive risk measures under ambiguous discount factors

Consider a regulator assessing the risk of a future payoff X_T when the discount factor D_T is ambiguous and ranges between two positive constants, $0 \leq d_L \leq D_T \leq d_H \leq 1$, according to her beliefs. The regulator is endowed with a spot risk measure ρ_0 and adverse to ambiguity on discount factor. Hence she assesses the risk of X_T in the interest rates worst case scenario

$$(3.1) \quad \mathcal{R}^{\rho_0}(X_T) := \sup_{D_T \in \mathcal{X}} \{ \rho_0(D_T X_T) \mid d_L \leq D_T \leq d_H \}.$$

Proposition 3.2 The functional \mathcal{R}^{ρ_0} in equation (3.1) is a cash sub-additive risk measure. \mathcal{R}^{ρ_0} can be rewritten as $\mathcal{R}^{\rho_0}(X_T) = \rho_0(-v(X_T))$, where $v(x) = -(d_L x^+ - d_H x^-)$ is convex decreasing function with left derivative v_x such that $v_x \in [-1, 0]$ and $x^+ = \sup(x, 0)$, $x^- = \sup(-x, 0)$.

Proof. \mathcal{R}^{ρ_0} is a cash sub-additive risk measure as it is the supremum of cash sub-additive, convex and monotone functions with respect to $X_T \in \mathcal{X}$. Moreover, since the $\inf_{D_T \in \mathcal{X}} \{D_T X_T | d_L \leq D_T \leq d_H\}$ is attained by $D_T^* = d_L 1_{\{X_T \geq 0\}} + d_H 1_{\{X_T < 0\}}$, then $\sup_{D_T \in \mathcal{X}} \{\rho_0(D_T X_T) | d_L \leq D_T \leq d_H\} = \rho_0(\inf_{D_T \in \mathcal{X}} \{D_T X_T | d_L \leq D_T \leq d_H\}) = \rho_0(d_L X_T^+ - d_H X_T^-)$, where $v(x) = -(d_L x^+ - d_H x^-)$. \square

Remark 3.3 When D_T varies between two random variables D_L and D_H in \mathcal{X} , $0 \leq D_L \leq D_T \leq D_H \leq 1$, the functional in (3.1) is a cash sub-additive risk measure $\mathcal{R}^{\rho_0}(X_T) = \rho_0(-V(X_T))$, where V is the random function $V(\omega, x) = -(D_L(\omega)x^+ - D_H(\omega)x^-)$, convex, decreasing with respect to x , $V_x \in [-1, 0]$, for any given $\omega \in \Omega$, and \mathcal{F}_T -measurable for any given $x \in \mathbb{R}$. Notice that when $D_L = D_T$, $\mathcal{R}^{\rho_0}(X_T) = \rho_0(D_T X_T)$.

Next example of cash sub-additive risk measure is not related to risky/ambiguous discount factor but to insurance risks. Following Jarrow (2002), the put option premium with zero strike price may be used as a possible measure of the firm insolvency risk. The expected losses are discounted using the risk free gross return $r \geq 1$.

Corollary 3.4 *Put premium risk measure. The premium of a put option with strike price zero and maturity T ,*

$$(3.2) \quad \mathcal{R}_p(X_T) := \frac{1}{r} \mathbb{E}_{\mathbb{P}}[(-X_T)^+],$$

is a coherent cash sub-additive risk measure as a function of the underlying asset price X_T .

Proof. The cash sub-additive risk measure in (3.1) coincides with the put option premium \mathcal{R}_p when $\rho_0(\cdot) = \mathbb{E}_{\mathbb{P}}[-(\cdot)]$, $d_L = 0$ and $d_H = 1/r$. \square

Remark 3.5 For any given strike price K the premium of a put option, $\mathcal{R}_p(X_T) := \frac{1}{r} \mathbb{E}_{\mathbb{P}}[(K - X_T)^+]$ is a cash sub-additive risk measure. This follows setting in equation (3.1) ρ_0 equals to the non normalized risk measure $\rho_0(X_T) = \mathbb{E}_{\mathbb{P}}[K - X_T]$ and $-v(x) = \frac{1}{r} \max(K - x, 0)$.

3.2.2 Composing cash additive risk measures and convex functions

Generalizing the previous examples we show that $\rho_0(-V)$ is a cash sub-additive risk measure, where V is a random function $V : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, $V(\omega, x)$, such that, for any $\omega \in \Omega$, $V(\omega, \cdot)$ is, lower-semicontinuous (lsc), decreasing, convex and $V(\omega, 0) = 0$, $V_x \in [-1, 0]$, and for any $x \in \mathbb{R}$, $V(\cdot, x)$ is \mathcal{F}_T -measurable. Moreover, $\rho_0(-V)$ can be represented in terms of finitely additive measures and \mathcal{F}_T -measurable “discount factors” over a set of possible scenarios that can be chosen according to the beliefs of the agent/regulator.

From standard results in convex analysis $V(\omega, x) = \sup_{y \in \mathbb{Q}} \{xy - \beta_T(\omega, y)\}$, where β_T is the random convex Fenchel transform of V , $\beta_T(\omega, y) := \sup_{x \in \mathbb{Q}} \{xy - V(\omega, x)\}$. Notice that β_T is finite only if $y \in [-1, 0]$ as $V_x \geq -1$. For example, the Fenchel transform of $v(x) = -(D_L x^+ - D_H x^-)$ is $\beta_T(y) = l^{\mathcal{D}}(-y)$, where $l^{\mathcal{D}}$ is the convex indicator function of the set $\mathcal{D} = [D_L, D_H]$, equal to 0 on \mathcal{D} and ∞ otherwise. While $V_x \geq -1$ is a necessary condition to obtain a cash sub-additive functional, the decreasing monotonicity ($V_x \leq 0$) and convexity of V insure the convexity and decreasing monotonicity of $\rho_0(-V)$.

Proposition 3.6 *Let V be a random, lsc, decreasing convex function as above and β_T the convex Fenchel transform of V . Let ρ_0 be a cash additive risk measure defined on \mathcal{X} with minimal penalty function α_0 . $\mathcal{R}^{\rho_0, V}(X_T) := \rho_0(-V(X_T))$ is a cash sub-additive risk measure, derived from the spot risk measure ρ_0 by assessing discount factors ambiguity through the penalty function β_T ,*

$$(3.3) \quad \mathcal{R}^{\rho_0, V}(X_T) = \sup_{D_T \in \mathcal{X}} \{ \rho_0(D_T X_T + \beta_T(-D_T)) \mid 0 \leq D_T \leq 1 \}.$$

Moreover, $\mathcal{R}^{\rho_0, V}(X_T) = \rho_0(-V(X_T))$ admits the dual representation

$$(3.4) \quad \mathcal{R}^{\rho_0, V}(X_T) = \sup_{Q_0 \in \mathcal{M}_{1,f}, D_T \in \mathcal{X}} \{ \mathbb{E}_{Q_0}[-D_T X_T] - \alpha^{\rho_0, V}(Q_0, D_T) \mid 0 \leq D_T \leq 1 \}$$

$$(3.5) \quad \alpha^{\rho_0, V}(Q_0, D_T) := \alpha_0(Q_0) + \mathbb{E}_{Q_0}[\beta_T(-D_T)].$$

For instance, if ρ_0 is the coherent worst case risk measure, that is, $\rho_0(X_T) = \rho_{\max}(X_T) = \sup_{Q_0 \in \mathcal{M}_1} \mathbb{E}_{Q_0}[-X_T]$, then $\mathcal{R}^{\rho_0, V}(X_T) = \rho_{\max}(-V(X_T)) = \| -V(X_T) \|_{\infty} = -V(\|X_T\|_{\infty})$ and $\alpha^{\rho_0, V}(Q_0, D_T) := \mathbb{E}_{Q_0}[\beta(-D_T)]$.

Remark 3.7 Representation (3.4)–(3.5) provides a better understanding of the different risks involved in the evaluation of the risky position X_T . The scenarios could be exogenously deter-

mined, for instance by some regulatory institution. The penalty function $\alpha^{\rho_0, V}$ depending on the ambiguous model and ambiguous discount factor could be determined by the preferences of the economic agent on Q_0 and D_T .

Remark 3.8 *Robust expected utilities and cash sub-additive risk measures.* By definition, the functional $\mathcal{R}^{\rho_0, V}$ admits a representation in terms of the ambiguous model and the convex function on the risky positions, $\mathcal{R}^{\rho_0, V}(X_T) = \sup_{Q_0 \in \mathcal{M}_{1,f}} \{\mathbb{E}_{Q_0}[V(X_T)] - \alpha_0(Q_0)\}$. The opposite of $\mathcal{R}^{\rho_0, V}$ can be viewed as examples of robust expected utilities associated with concave functions $U = -V$ and concave penalty functions $\tilde{\alpha}_0 = -\alpha_0$, that is $-\mathcal{R}^{\rho_0, V}(X_T) = \inf_{Q_0 \in \mathcal{M}_{1,f}} \{\mathbb{E}_{Q_0}[U(X_T)] - \tilde{\alpha}_0(Q_0)\}$. Notice that U does not satisfy the Inada conditions. For robust expected utilities see, for instance, Schied (2004) and Maccheroni, Marinacci, and Rustichini (2004).

Proof. Firstly we prove that $\mathcal{R}^{\rho_0, V}$ is a cash sub-additive risk measure. *Decreasing monotonicity:* The increasing monotonicity of $-V$ and the decreasing monotonicity of ρ_0 imply the decreasing monotonicity of $\mathcal{R}^{\rho_0, V}$.

Convexity: The concavity of $-V$, the decreasing monotonicity and the convexity of ρ_0 imply the convexity of $\mathcal{R}^{\rho_0, V}$.

Cash sub-additivity: $\mathcal{R}^{\rho_0, V}(X_T + m) + m = \rho_0(-V(X_T + m1_T)) + m = \rho_0(-V(X_T + m1_T) - m)$ is increasing in m if $-V(X_T + m1_T) - m$ is decreasing in m . As $V_x > -1$ the result follows.

Representations: To prove (3.3) we observe that

$$\rho_0(-V(X_T)) = \rho_0\left(\inf_{-1 \leq y \leq 0} \{-X_T y + \beta_T(y)\}\right) = \rho_0\left(\inf_{D_T \in \mathcal{X}} \{D_T X_T + \beta_T(-D_T) \mid 0 \leq D_T \leq 1\}\right).$$

From the decreasing monotonicity of ρ_0 , for any $\tilde{D}_T \in \mathcal{X}$, $0 \leq \tilde{D}_T \leq 1$ we have

$$(3.6) \quad \rho_0\left(\inf_{D_T \in \mathcal{X}} \{D_T X_T + \beta_T(-D_T) \mid 0 \leq D_T \leq 1\}\right) \geq \rho_0\left(\tilde{D}_T X_T + \beta_T(-\tilde{D}_T)\right).$$

The result follows setting $\tilde{D}_T = D_T^*$ in equation (3.6), where $D_T^* \in \mathcal{X}$ is the element achieving the $\inf_{D_T \in \mathcal{X}} \{D_T X_T + \beta_T(-D_T) \mid 0 \leq D_T \leq 1\}$. Finally, representations (3.4)–(3.5) are obtained from the dual representation of ρ_0 and from (3.3). \square

The penalty function $\alpha^{\rho_0, v}$ in (3.5) is not the minimal one. As any pair (Q_0, D_T) defines a unique

additive set function μ absolutely continuous with respect to Q_0 , $d\mu := D_T dQ_0$, $0 \leq \mu(\Omega) \leq 1$, the functional $\mathcal{R}^{\rho_0, V}$ can be rewritten as

$$\mathcal{R}^{\rho_0, V}(X_T) = \sup_{\mu \in \mathcal{M}_{1, f}(\mathcal{F}_T)} \left\{ \mu(-X_T) - \gamma(\mu) \mid 0 \leq \mu(\Omega) \leq 1 \right\},$$

where $\mu(-X_T) := \int -X_T(\omega) \mu(d\omega)$ and $\gamma(\mu) = \inf_{Q_0 \in \mathcal{M}_{1, f}} \left\{ \alpha_0(Q_0) + \mathbb{E}_{Q_0} \left[\beta_T \left(-\frac{d\mu}{dQ_0} \right) \right] \right\}$ for any μ such that $d\mu = D_T dQ_0$, $0 \leq D_{0, T} \leq 1$, and $\gamma = \infty$ otherwise.

Next section gives the dual representation of the cash sub-additive risk measures \mathcal{R} in terms of the minimal penalty function.

4 Minimal cash additive extension of \mathcal{R} and duality

In this section we study the dual representation of cash sub-additive risk measures. To obtain duality results we can either use convex analysis tools (for instance adapting the techniques for convex risk measures in Krätschmer (2007)), or recover cash sub-additive risk measures by enlarging the space of risky positions. We adopt the second approach because of its richer financial interpretation, despite the fact that the first one could be less involved. Our approach provides an interesting interpretation of cash additive and cash sub-additive risk measures where default events or stochastic numéraires are taken into account. Taking a classical procedure in credit risk modeling, we consider a minimal enlargement of the sample space Ω and we extend the cash sub-additive risk measure \mathcal{R} into a cash additive risk measure, which is in a one to one correspondence with \mathcal{R} . This allows to derive properties of \mathcal{R} and the dual representation using classical theory on cash additive risk measures.

Cheridito and Kupper (2006) use a similar procedure to decompose dynamic cash additive risk measures in one-step generators and to provide a dual representation of these generators. Interestingly, these generators are cash sub-additive risk measures with opposite sign³.

While in the dual representation of cash additive risk measures the set functions Q are normalized to one in $\mathcal{M}_{1, f}(\mathcal{F}_T)$, the dual representation of cash sub-additive risk measures is based on finite additive set functions μ with total mass $0 \leq \mu(\Omega) \leq 1$, called sub-probability measures, whose set is denoted by $\mathcal{M}_{s, f}(\mathcal{F}_T)$. A simple procedure to obtain a cash additive risk measure using a cash sub-additive risk measure \mathcal{R} is as follows. While \mathcal{R} is not cash additive with respect to $X_T \in \mathcal{X}$, the bivariate function $\hat{\rho}(X_T, x) := \mathcal{R}(X_T 1_T - x 1_T) - x$ as a function

³We thank the referee for highlighting this result.

of the pair (X_T, x) is cash additive. In the sequel, we introduce the minimal measurable space where the pair (X_T, x) is the coordinate of a random variable and $\hat{\rho}$ is a cash additive risk measure on these random variables.

4.1 Minimal extension of \mathcal{R} into a cash additive risk measure

Any pair (X_T, x) where $X_T \in \mathcal{X}$ and $x \in \mathbb{R}$ can be viewed as the coordinates of a function defined on the enlarged space $\hat{\Omega} = \Omega \times \{0, 1\}$ with element (ω, θ) ,

$$\hat{X}_T(\omega, \theta) := X_T(\omega)1_{\theta=1} + x1_{\theta=0}.$$

$\hat{\Omega}$ is endowed with the σ -algebra $\hat{\mathcal{F}}_T$ generated by the bounded random variables \hat{X}_T . Notice that $\hat{\mathcal{F}}_T$ is not the product σ -algebra. Let $\hat{\mathcal{X}}$ be the linear space of all bounded random variables \hat{X}_T . To denote $\hat{X}_T \in \hat{\mathcal{X}}$ we use its coordinates $\hat{X}_T = (X_T, x)$. The constant variables are denoted by $\hat{m} = (m, m)$ and $\hat{m} = m1_{\theta=1} + m1_{\theta=0} = m$. The event $\{\theta = 0\}$ is atomic and all $\hat{\mathcal{F}}_T$ -random variables are constant on this event. The event $\{\theta = 1\}$ models the risk affecting the numéraire 1_T . Intuitively, θ is associated with the default time τ of the counterpart. The event $\{\theta = 1\}$ is equivalent to $\{\tau > T\}$. The choice of the atomic σ -algebra $\hat{\mathcal{F}}_T$ implies a one to one correspondence between normalized additive set function \hat{Q} in $\mathcal{M}_{1,f}(\hat{\mathcal{F}}_T)$ and sub-probability set functions in $\mathcal{M}_{s,f}(\mathcal{F}_T)$ on (Ω, \mathcal{F}_T) . Indeed, any \hat{Q} in $\mathcal{M}_{1,f}(\hat{\mathcal{F}}_T)$ can be decomposed as follows, $\forall \hat{X}_T = (X_T, x) \in \hat{\mathcal{X}}$,

$$(4.1) \quad \hat{Q}(\hat{X}_T) = \hat{Q}(X_T 1_{\theta=1}) + x\hat{Q}(1_{\theta=0}) = \mu(X_T) + x(1 - \mu(1)),$$

where $\mu(\cdot) := \hat{Q}(\cdot 1_{\theta=1})$ is an additive sub-probability of $\mathcal{M}_{s,f}(\mathcal{F}_T)$.

The following proposition shows how to extend the cash sub-additive functional \mathcal{R} into a cash additive risk measure $\hat{\rho}$ on $\hat{\mathcal{X}}$ and the one to one correspondence.

Proposition 4.1 *1) A normalized cash sub-additive risk measure \mathcal{R} on \mathcal{X} defines a normalized cash additive risk measure $\hat{\rho}$ on $\hat{\mathcal{X}}$,*

$$(4.2) \quad \forall \hat{X}_T = (X_T, x) \in \hat{\mathcal{X}}, \quad \hat{\rho}(\hat{X}_T) := \hat{\rho}((X_T, x)) := \mathcal{R}(X_T - x1_T) - x.$$

Notice that $\hat{\rho}(X_T 1_{\theta=1}) = \mathcal{R}(X_T)$.

2) Any cash additive risk measure on $\hat{\mathcal{X}}$ restricted to the event $\{\theta = 1\}$ defines a cash sub-additive risk measure which satisfies equation (4.2).

Remark 4.2 The cash sub-additive risk measure \mathcal{R} can be used to measure the risk of default-able contingent claims \widehat{X}_T when there is no compensation ($x = 0$) if the default occurs, $\{\theta = 0\}$.

The proof relies on the cash sub-additive property to obtain a monotone decreasing functional.

Proof. 1) *Cash additive:* Let $\widehat{X}_T = (X_T, x) \in \widehat{\mathcal{X}}$ and $m \in \mathbb{R}$. By definition, $\hat{\rho}(X_T 1_{\theta=1} + x 1_{\theta=0} + m 1_{\theta=1} + m 1_{\theta=0}) = \mathcal{R}(X_T + m 1_T - (x + m) 1_T) - (x + m) = \hat{\rho}(\widehat{X}_T) - m$.

Decreasing monotonicity: Let $\widehat{X}_T = (X_T, x)$ and $\widehat{Y}_T = (Y_T, y) \in \widehat{\mathcal{X}}$ such that $\widehat{X}_T \geq \widehat{Y}_T$, that is $X_T \geq Y_T$ and $x \geq y$. From the cash sub-additivity and the decreasing monotonicity of \mathcal{R} it follows that $\hat{\rho}(\widehat{X}_T) = \mathcal{R}(X_T - x 1_T) - x \leq \mathcal{R}(X_T - y 1_T) - y \leq \mathcal{R}(Y_T - y 1_T) - y = \hat{\rho}(\widehat{Y}_T)$.

Convexity: We use the notation in equation (2.1). From the convexity of \mathcal{R} , $\mathcal{R}(\text{Bar}[X_I]) \leq \text{Bar}[\mathcal{R}(X)_I]$. This implies that $\hat{\rho}(\text{Bar}[\widehat{X}_I]) = \mathcal{R}(\text{Bar}[X_I - x_I]) - \text{Bar}[x_I] \leq \text{Bar}[\mathcal{R}(X - x)_I] - \text{Bar}[x_I] = \text{Bar}[\hat{\rho}(\widehat{X})_I]$, which shows the convexity of $\hat{\rho}$.

2) Let $\check{\rho}$ be a cash additive risk measure on $\widehat{\mathcal{X}}$. We have to show that $\mathcal{R}^{\check{\rho}}(X_T) := \check{\rho}(X_T 1_{\theta=1})$ is a cash sub-additive risk measure. The decreasing monotonicity and convexity follow from the definition. The cash sub-additive property is verified observing that $\mathcal{R}^{\check{\rho}}(X_T + m 1_T) + m = \hat{\rho}((X_T + m) 1_{\theta=1}) + m = \hat{\rho}(X_T 1_{\theta=1} - m 1_{\theta=0})$ is increasing in m . \square

4.2 Dual representation of cash sub-additive risk measures

In the next proposition we use the one to one correspondence in equation (4.2) between $\hat{\rho}$ and \mathcal{R} to characterize cash sub-additive risk measures. We show that the minimal penalty function of \mathcal{R} and the minimal penalty function $\hat{\rho}$ coincide and are concentrated on the set of sub-probability measures $\mathcal{M}_{s,f}(\mathcal{F}_T)$. Moreover, under the additional assumption of continuity from below of \mathcal{R} the dual representation in terms of σ -additive sub-probability measures is obtained.

Theorem 4.3 (a) *Any cash sub-additive risk measure \mathcal{R} on \mathcal{X} can be represented in terms of finitely additive sub-probability measures,*

$$(4.3) \quad \forall X_T \in \mathcal{X}, \quad \mathcal{R}(X_T 1_T) = \sup_{\mu \in \mathcal{M}_{s,f}(\mathcal{F}_T)} \{ \mu(-X_T) - \alpha^{\mathcal{R}}(\mu) \}, \quad \alpha^{\mathcal{R}}(\mu) := \hat{\alpha}(\widehat{Q}),$$

where $\mu(\cdot) = \widehat{Q}(\cdot 1_{\theta=1})$ and $\hat{\alpha}$ is any penalty function representing $\hat{\rho}$. In particular, if $\hat{\alpha}$ is the minimal penalty function for $\hat{\rho}$ then $\alpha^{\mathcal{R}}$ is the minimal penalty function for \mathcal{R} and $\alpha^{\mathcal{R}}(\mu) = \sup_{X_T \in \mathcal{X}} \{ \mu(-X_T) - \mathcal{R}(X_T) \}$.

(b) When \mathcal{R} is a cash sub-additive risk measure continuous from below any penalty function β representing \mathcal{R} is concentrated on the class $\mathcal{M}_s(\mathcal{F}_T)$ of σ -additive sub-probability measures, i.e., $\beta(\mu) < \infty \Rightarrow \mu$ is σ -additive.

Proof. (a) From Proposition 4.1, $\mathcal{R}(X_T 1_T) = \hat{\rho}(X_T 1_{\theta=1})$. Equation (4.3) is implied by the dual representation of $\hat{\rho}$ and the one to one correspondence between \hat{Q} and μ : $\hat{Q}(\cdot 1_{\theta=1}) = \mu(\cdot)$. Let $\hat{\alpha}$ be the minimal penalty function of $\hat{\rho}$. By definition of the minimal penalty function,

$$\begin{aligned}
\hat{\alpha}(\hat{Q}) &= \sup_{\hat{X}_T \in \hat{\mathcal{X}}_T} \{ \mathbb{E}_{\hat{Q}}[-X_T 1_{\theta=1} - x 1_{\theta=0}] - \hat{\rho}(\hat{X}_T) \} \\
&= \sup_{\hat{X}_T \in \hat{\mathcal{X}}} \{ \mathbb{E}_{\hat{Q}}[-(X_T - x) 1_{\theta=1}] - x - \mathcal{R}(X_T - x 1_T) + x \} \\
(4.4) \quad &= \sup_{X_T \in \mathcal{X}_T} \{ \mathbb{E}_{\hat{Q}}[-(X_T) 1_{\theta=1}] - \mathcal{R}(X_T) \}, \quad \hat{Q} \in \mathcal{M}_{1,f}(\hat{\mathcal{F}}_T).
\end{aligned}$$

As $\hat{Q}(\cdot 1_{\theta=1}) = \mu(\cdot)$, from equation (4.4) we have $\alpha^{\mathcal{R}}(\mu) := \hat{\alpha}(\hat{Q}) = \sup_{X_T \in \mathcal{X}} \{ \mu(-X_T) - \mathcal{R}(X_T) \}$, showing that $\alpha^{\mathcal{R}}$ is the minimal penalty function for \mathcal{R} .

(b) If \mathcal{R} is continuous from below the cash additive $\hat{\rho}$ is continuous from below as a function of $\hat{X}_T = (X_T, x)$. Then from Theorem 2.2 follows that the penalty function of $\hat{\rho}$ is concentrated on the class $\mathcal{M}_1(\hat{\mathcal{F}}_T)$. This implies that the penalty function of \mathcal{R} is concentrated on the set of σ -additive sub-probability $\mathcal{M}_s(\mathcal{F}_T)$. \square

Next corollary shows a representation of \mathcal{R} where the penalty functional depends on constants $c \in [0, 1]$ and probability measures. Frittelli and Scandolo (2006) provide examples of general capital requirement with similar representations. It is interesting to observe that, among these, the only capital requirement that satisfies the property of cash sub-additivity is the one reflecting the agent's temporal risk aversion, which is related to the uncertainty in the numéraire. For more details see Section 6 in Frittelli and Scandolo (2006).

Corollary 4.4 *Any cash sub-additive risk measure \mathcal{R} can be represented as follows*

$$(4.5) \quad \forall X_T \in \mathcal{X}, \quad \mathcal{R}(X_T 1_T) = \sup_{(c, Q_T) \in [0, 1] \times \mathcal{M}_{1,f}(\mathcal{F}_T)} \{ c \mathbb{E}_{Q_T}(-X_T) - \alpha^{\mathcal{R}}(c \cdot Q_T) \}.$$

When \mathcal{R} is continuous from below, the penalty function $\alpha^{\mathcal{R}}(c \cdot Q_T)$ is concentrated on the set $[0, 1] \times \mathcal{M}_1(\mathcal{F}_T)$ where $\mathcal{M}_1(\mathcal{F}_T)$ is set of σ -additive sub-probability. If $\inf_{X_T \in \mathcal{X}} \mathcal{R}(X_T) = -\infty$, the constants c in formula (4.5) are strictly positive.

Proof. Equation (4.5) follows by normalizing the sub-probability in equation (4.3), more precisely defining, for any $\mu \in \mathcal{M}_{s,f}$ such that $\mu > 0$, $Q_T(\cdot) = \mu(\cdot)/c$ where $c := \mu(\Omega)$. If $\mu = 0$ then $c = 0$ and $Q_T \in \mathcal{M}_{1,f}(\mathcal{F}_T)$ is not uniquely identified. The condition $\inf_{X_T \in \mathcal{X}} \mathcal{R}(X_T) = -\infty$ implies $-\alpha^{\mathcal{R}}(0) = \inf_{X_T \in \mathcal{X}} \mathcal{R}(X_T) = -\infty$ excluding $\mu = 0$. \square

The following representation, suggested by an anonymous referee, provides a characterization of cash sub-additive risk measures in terms of ambiguous “zero-coupon bond” viewed as a deterministic discount factor. The risky positions are assessed via a family of forward convex risk measures (see Definition 2.6) parameterized by the deterministic discounted factors⁴.

Corollary 4.5 *Any cash sub-additive risk measure \mathcal{R} such that $\inf_{X_T \in \mathcal{X}} \mathcal{R}(X_T) = -\infty$ is a worst discounted forward risk measures, that is*

$$(4.6) \quad \mathcal{R}(X_T 1_T) = \sup_{c \in (0,1]} c \cdot \rho_{T,c}(-X_T),$$

where $(\rho_{T,c})$ is a family of forward cash additive risk measures such that the functional $(X_T, c) \in \mathcal{X} \times (0,1] \rightarrow c \cdot \rho_{T,c}(-X_T)$ is convex.

Proof. Since any functional on the right side of equation (4.6) is a cash-sub additive, convex and monotone functional, their supremum shares the same property.

Vice versa, given a cash sub-additive risk measure \mathcal{R} and his dual representation, as $\inf_{X_T \in \mathcal{X}} \mathcal{R}(X_T) = -\infty$, we can define the forward risk measures,

$$\rho_{T,c}(-X_T) = \sup_{Q_T \in \mathcal{M}_{1,f}(\mathcal{F}_T)} \left\{ \mathbb{E}_{Q_T}(-X_T) - \frac{\alpha^{\mathcal{R}}(c \cdot Q_T)}{c} \right\}.$$

By definition, the family $c \cdot \rho_{T,c}(-X_T)$ is convex in both arguments (c, X_T) and \mathcal{R} can be rewritten as in equation (4.6). \square

5 Other cash additive extensions of \mathcal{R}

In this section we provide a representations of a cash sub-additive risk measures \mathcal{R} where the ambiguous discount factor/numéraire is explicitly modeled as random variables of \mathcal{X} . While for the cash sub-additive risk measures generated via convex functions (in Section 3.2) these

⁴We thank the referee for this stimulating suggestion.

representations were easily obtained, to derive similar results for a generic \mathcal{R} new assumptions and more involved techniques are required. To achieve this goal, we apply the same procedure as in Section 4 and we extend \mathcal{R} to a larger space that contains \mathcal{X} . In this case the extension is not unique and requires the introduction of an auxiliary a priori cash additive risk measure. Then, for the cash sub-additive risk measures generated via convex functions we propose another extension on the same enlarged space obtained through a conditional risk measure.

5.1 Cash sub-additive risk measures and ambiguous discounted factors

To define a linear space which contains \mathcal{X} , the σ -algebra $\widehat{\mathcal{F}}_T$ defined in Section 4 is replaced by the product σ -algebra \mathcal{G}_T . On $(\Omega \times \{0, 1\}, \mathcal{G}_T)$ any bounded \mathcal{G}_T -random variable \tilde{X}_T can be represented as $\tilde{X}_T(\omega, \theta) = X_T^1(\omega)1_{\theta=1} + X_T^0(\omega)1_{\theta=0}$ and $X_T^0, X_T^1 \in \mathcal{X}$. Let $\tilde{\mathcal{X}}$ be the linear space of all the bounded \mathcal{G}_T -random variables \tilde{X}_T . We refer to \tilde{X}_T using the short notation $\tilde{X}_T = (X_T^1, X_T^0)$. The diagonal elements $\tilde{X}_T = (X_T, X_T)$ coincide with X_T and the corresponding σ -algebra with \mathcal{F}_T . This identification was not possible for the random variables $\hat{X} = (X_T, x)$ defined in the previous section.

Now we discuss the probabilistic structure of $(\Omega \times \{0, 1\}, \mathcal{G}_T)$. Notice that in this section we consider probability measures and not finite additive set functions.

Definition 5.1 For any given probability measure $\tilde{\mathbb{Q}} \in \mathcal{M}_1(\mathcal{G}_T)$ let \mathbb{Q} denote the restriction of $\tilde{\mathbb{Q}}$ to \mathcal{F}_T , $\mathbb{Q} := \tilde{\mathbb{Q}}|_{\mathcal{F}_T}$, and $D_T \in [0, 1]$ the \mathcal{F}_T -conditional probability of the event $\{\theta = 1\}$, $D_T := \mathbb{E}_{\tilde{\mathbb{Q}}}[1_{\theta=1}|\mathcal{F}_T]$, also called discount factor. We denote $\overline{\mathbb{Q}}$ the probability measure associated with the restriction of $\tilde{\mathbb{Q}}$ to the event $\{\theta = 0\}$, which is uniquely determined by (\mathbb{Q}, D_T)

$$(5.1) \quad \mathbb{Q}(X_T) = \mathbb{Q}(D_T X_T) + (1 - \mathbb{Q}(D_T))\overline{\mathbb{Q}}(X_T).$$

$\overline{\mathbb{Q}}$ is a probability measure absolutely continuous with respect to \mathbb{Q} , with Radon-Nikodym density given by $\overline{\Delta}_T := \frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} = \frac{1-D_T}{(1-\mathbb{Q}(D_T))}$, $0 \leq \overline{\Delta}_T \leq 1$, $\mathbb{Q}(\overline{\Delta}_T) = 1$.

For any $\tilde{X}_T = X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0} \in \tilde{\mathcal{X}}$,

$$(5.2) \quad \tilde{\mathbb{Q}}(X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0}) = \mathbb{Q}(X_T^1 D_T) + \mathbb{Q}(X_T^0 (1 - D_T)) = \mathbb{Q}(X_T^1 D_T) + (1 - \mathbb{Q}(D_T))\overline{\mathbb{Q}}(X_T^0).$$

Remark 5.2 *The interpretation of D_T in credit risk.* In credit risk, θ is associated with the default time of the counterpart τ , where τ is a positive random variable non \mathcal{F}_T -measurable. The event $\{\theta = 1\}$ can be viewed as $\{\tau > T\}$ and $\mathbb{E}_{\tilde{\mathbb{Q}}}[1_{\theta=1}|\mathcal{F}_T]$ as the conditional survival probability function of τ at time T . $\tilde{X}_T = X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0} \in \tilde{\mathcal{X}}$ is a defaultable contingent claim that pays X_T^1 (at time T) if there is no default ($\tau > T$) and X_T^0 otherwise.

In the sequel we extend \mathcal{R} into a cash additive risk measure $\tilde{\rho}$ on the enlarged space $\tilde{\mathcal{X}}$. Via the penalty function of $\tilde{\rho}$, a representation of cash sub-additive risk measures will be given in terms of the ambiguous probability measure and the ambiguous discount factor, both on the original space of definition of \mathcal{R} . To define this cash additive risk measure on $\tilde{\mathcal{X}}$ we use, as in Section 4.1, the cash additive risk measure $\hat{\rho}$ in (4.2). In this case $\tilde{X}_T = (X_T^1, X_T^0) \in \tilde{\mathcal{X}}$ has two risky components and we introduce an a priori risk measure $\bar{\rho}$ assessing the risk of the second component.

Definition 5.3 Let \mathcal{R} be a cash sub-additive risk measure and $\bar{\rho}$ a cash additive risk measure both normalized and defined on \mathcal{X} . The functional on $\tilde{\mathcal{X}}$

$$(5.3) \quad \tilde{\rho}(\tilde{X}_T) = \tilde{\rho}(X_T^1, X_T^0) := \mathcal{R}(X_T^1 + \bar{\rho}(X_T^0)1_T) + \bar{\rho}(X_T^0) = \hat{\rho}(X_T^1, -\bar{\rho}(X_T^0))$$

and its restriction on \mathcal{X} ,

$$(5.4) \quad \rho_{\mathcal{R}, \bar{\rho}}(X_T) := \mathcal{R}(X_T + \bar{\rho}(X_T)1_T) + \bar{\rho}(X_T) = \hat{\rho}(X_T, -\bar{\rho}(X_T)),$$

are cash additive risk measures. Moreover, $\tilde{\rho}(X_T 1_{\theta=1}) = \mathcal{R}(X_T 1_T)$.

The following theorem shows that \mathcal{R} can be written as a function of probability measures $\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T)$ and \mathcal{F}_T -measurable discount factors $D_T \in \mathcal{X}$ using the minimal penalty function of the cash additive risk measure $\tilde{\rho}$. This representation is similar to the dual representation (see equations (3.4)–(3.5)) of cash sub-additive risk measures generated by convex functions.

We consider penalty functions concentrated on the class of probabilities measures assuming that \mathcal{R} and $\bar{\rho}$ are continuous from below. This implies that also $\tilde{\rho}$ and $\rho_{\mathcal{R}, \bar{\rho}}$ are continuous from below.

Theorem 5.4 Assume that the convex functionals \mathcal{R} and $\bar{\rho}$ are continuous from below. Let $\alpha^{\mathcal{R}}$ and $\bar{\alpha}$ be the minimal penalty functions of \mathcal{R} and $\bar{\rho}$, respectively. Let $\tilde{\alpha}$ be the minimal penalty function of $\tilde{\rho}$ defined in equation (5.3). For any $\tilde{\mathbb{Q}} \in \mathcal{M}_1(\mathcal{G}_T)$, let \mathbb{Q} , D_T and $\bar{\mathbb{Q}}$ be as in Definition 5.1, such that $\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} = \bar{\Delta}_T = \frac{1 - D_T}{(1 - \mathbb{Q}(D_T))}$.

1) The cash sub-additive risk measure \mathcal{R} can be represented as

$$(5.5) \quad \mathcal{R}(X_T 1_T) = \tilde{\rho}(X_T 1_{\theta=1}) = \sup_{\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T), D_T \in [0,1]} \{ \mathbb{E}_{\mathbb{Q}}(-D_T X_T) - \tilde{\alpha}(D_T, \mathbb{Q}) \},$$

where the minimal penalty $\tilde{\alpha}$ has the following form

$$(5.6) \quad \tilde{\alpha}(\tilde{\mathbb{Q}}) = \tilde{\alpha}(\mathbb{Q}, D_T) = \alpha^{\mathcal{R}}(D_T \cdot \mathbb{Q}) + (1 - \mathbb{Q}(D_T)) \bar{\alpha}(\bar{\mathbb{Q}}), \quad \tilde{\mathbb{Q}} \in \mathcal{M}_1(\mathcal{G}_T).$$

Notice that $\tilde{\mathbb{Q}} \in \text{Dom}(\tilde{\alpha})$ if and only if $\mathbb{Q} \cdot D_T \in \text{Dom}(\alpha^{\mathcal{R}})$ and $\bar{\mathbb{Q}} \in \text{Dom}(\bar{\alpha})$.

2) The minimal penalty function of $\rho_{\mathcal{R}, \bar{\rho}}$ in equation (5.4) is given by, for any $\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T)$,

$$(5.7) \quad \alpha_{\mathcal{R}, \bar{\rho}}(\mathbb{Q}) = \inf_{D_T, \bar{\mathbb{Q}}} \{ \alpha^{\mathcal{R}}(D_T \cdot \mathbb{Q}) + (1 - \mathbb{Q}(D_T)) \bar{\alpha}(\bar{\mathbb{Q}}) \mid \mathbb{Q}(\cdot) = \mathbb{Q}(D_T \cdot) + (1 - \mathbb{Q}(D_T)) \bar{\mathbb{Q}}(\cdot) \}.$$

Remark 5.5 When \mathcal{R} and $\bar{\rho}$ are both coherent risk measures, equation (5.5) reduces to

$$\mathcal{R}(X_T) = \tilde{\rho}(X_T 1_{\theta=1}) = \sup_{\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T), D_T \in [0,1]} \{ \mathbb{E}_{\mathbb{Q}}(-D_T X_T) \mid D_T \cdot \mathbb{Q} \in \text{Dom}(\alpha^{\mathcal{R}}), \bar{\Delta}_T \cdot \mathbb{Q} \in \text{Dom}(\bar{\alpha}) \}.$$

Proof. 1) The representation (5.5) of \mathcal{R} follows from $\mathcal{R}(X_T 1_T) = \tilde{\rho}(X_T 1_{\theta=1})$ and equation (5.2).

To obtain the decomposition of the minimal penalty function in equation (5.6) we use the representation of $\tilde{\mathbb{Q}}$ in terms of $\mathbb{Q}(D_T \cdot)$ and $\bar{\mathbb{Q}}$ given in definition 5.1. From the definition of $\tilde{\rho}$ and of the minimal penalty function we have

$$\begin{aligned} \tilde{\alpha}(\tilde{\mathbb{Q}}) &= \sup_{(X_T^1, X_T^0) \in \tilde{\mathcal{X}}} \left\{ \tilde{\mathbb{Q}}(-X_T^1 1_{\theta=1} - X_T^0 1_{\theta=0}) - \mathcal{R}(X_T^1 + \bar{\rho}(X_T^0) 1_T) - \bar{\rho}(X_T^0) \right\} \\ &= \sup_{(X_T^1, X_T^0) \in \tilde{\mathcal{X}}} \left\{ \tilde{\mathbb{Q}}(-(X_T^1 + \bar{\rho}(X_T^0)) 1_{\theta=1}) - \mathcal{R}(X_T^1 + \bar{\rho}(X_T^0) 1_T) + \tilde{\mathbb{Q}}(-(X_T^0 + \bar{\rho}(X_T^0)) 1_{\theta=0}) \right\}. \end{aligned}$$

Using the change of variable $Y_T := X_T^1 + \bar{\rho}(X_T^0)$ and equations (5.1)–(5.2) give the result

$$\begin{aligned} \tilde{\alpha}(\tilde{\mathbb{Q}}) &= \sup_{(Y_T, X_T^0)} \left\{ \mathbb{Q}(-Y_T D_T) - \mathcal{R}(Y_T) + (1 - \mathbb{Q}(D_T)) [\bar{\mathbb{Q}}(-(X_T^0 + \bar{\rho}(X_T^0)))] \right\} \\ &= \alpha^{\mathcal{R}}(D_T \cdot \mathbb{Q}) + (1 - \mathbb{Q}(D_T)) \sup_{X \in \mathcal{A}^{\bar{\rho}}} \{ \mathbb{Q}(-X \bar{\Delta}_T) \} \\ &= \alpha^{\mathcal{R}}(D_T \cdot \mathbb{Q}) + (1 - \mathbb{Q}(D_T)) \bar{\alpha}(\bar{\Delta}_T \cdot \mathbb{Q}). \end{aligned}$$

2) To obtain the penalty function $\alpha_{\mathcal{R},\bar{\rho}}$ of $\rho_{\mathcal{R},\rho_0}$ we restrict $\tilde{\rho}$ on \mathcal{F}_T and we use equation (5.1)

$$\begin{aligned}\rho_{\mathcal{R},\rho_0}(X_T) &= \sup_{\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T)} \left\{ \mathbb{Q}(-X_T D_T) + (1 - \mathbb{Q}(D_T)) \bar{\mathbb{Q}}(-X_T) \right. \\ &\quad \left. - (\alpha^{\mathcal{R}}(D_T \cdot \mathbb{Q}) + (1 - \mathbb{Q}(D_T)) \bar{\alpha}(\bar{\mathbb{Q}})) \right\} \\ &= \sup_{\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T)} \left\{ \mathbb{Q}(-X_T) - (\alpha^{\mathcal{R}}(D_T \cdot \mathbb{Q}) + (1 - \mathbb{Q}(D_T)) \bar{\alpha}(\bar{\mathbb{Q}})) \right\}.\end{aligned}$$

Observing that for a given $\mathbb{Q} \in \mathcal{M}_1(\mathcal{F}_T)$ more then one pair $(D_T, \bar{\mathbb{Q}})$, $D_T \in \mathcal{X}$, $D_T \in [0, 1]$, can verify $\mathbb{Q}(-X_T D_T) + (1 - \mathbb{Q}(D_T)) \bar{\mathbb{Q}}(-X_T) = \mathbb{Q}(X_T)$ yields the equation (5.7). Similar calculations show that $\alpha_{\mathcal{R},\bar{\rho}}$ is the minimal penalty function. \square

5.2 Conditional risk measures and extensions on $\tilde{\mathcal{X}}$

This section reinterprets the cash sub-additive risk measures $\mathcal{R}^{\rho,V} = \rho(-V)$ studied in Section 3.2.2. These risk measures are now represented as the composition of the unconditional cash additive risk measure ρ and the *conditional* cash additive risk measure generated by the random function V . We obtain the result introducing a more natural extension of $\mathcal{R}^{\rho,V}$ called $\check{\rho}^V$ to the enlarged space $\tilde{\mathcal{X}}$. The restriction of $\check{\rho}^V$ to the space \mathcal{X} is ρ itself, and $\check{\rho}^V$ can be obtained composing ρ with a cash additive conditional risk measures. Moreover, we show that any cash additive risk measure on $\tilde{\mathcal{X}}$ generated from ρ via a conditional cash additive risk measure is associated to a cash sub-additive risk measure generated by a convex function.

As in Section 3.2.2, in the sequel ρ denotes a normalized cash additive risk measure and $V(\omega, x)$ an \mathcal{F}_T -measurable random functional convex monotone decreasing such that $V(0) = 0$ and with left derivative $V_x \in [-1, 0]$. From Proposition 3.6 we know that $\mathcal{R}^{\rho,V}(X_T) := \rho(-V(X_T))$ is a cash sub-additive risk measure on \mathcal{X} .

Proposition 5.6 *On the enlarged space $\tilde{\mathcal{X}}$ any cash additive risk measure ρ and any random function V define a cash additive risk measure,*

$$(5.8) \quad \check{\rho}^V(X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0}) := \rho(-V(X_T^1 - X_T^0) + X_T^0), \quad X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0} \in \tilde{\mathcal{X}}.$$

$\check{\rho}^V$ coincides with $\mathcal{R}^{\rho,V}$ on $\{\theta = 1\}$ and with ρ on $\mathcal{X} \subset \tilde{\mathcal{X}}$:

$$\check{\rho}^V(X_T 1_{\theta=1}) = \rho(-V(X_T)) = \mathcal{R}^{\rho,V}(X_T) \quad \text{and} \quad \check{\rho}^V((X_T, X_T)) = \rho(X_T).$$

Requiring V decreasing monotone and such that $V_x \in [-1, 0]$ is crucial to obtain $\check{\rho}^V$ decreasing monotone (see proof below).

Proof. Decreasing monotonicity: $\check{\rho}^V$ is decreasing monotone if $V(X_T^1 - X_T^0) - X_0$ is decreasing monotone with respect to (X_T^1, X_T^0) . Let $\tilde{X}_T = (X_T^1, X_T^0) \geq \tilde{Y}_T = (Y_T^1, Y_T^0)$, that is $X_T^1 \geq Y_T^1$ and $X_T^0 \geq Y_T^0$. As $V(x + m) + m$ is not decreasing in m , $V(X_T^1 - X_T^0) - X_T^0$ is not increasing in X_T^0 , then $V(X_T^1 - X_T^0) - X_T^0 \leq V(X_T^1 - Y_T^0) - Y_T^0 \leq V(Y_T^1 - Y_T^0) - Y_T^0$, where the last inequality is due to the decreasing monotonicity of V .

Cash additivity and *convexity* follow from the definition of $\check{\rho}^V$. \square

Now we recall the definition of conditional risk measures that in our setting⁵ reads as follows.

Definition 5.7 1) A *cash additive conditional risk measure* on \mathcal{F}_T is a monotone decreasing convex functional, $\tilde{\rho}_{\mathcal{F}_T} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ which satisfies the \mathcal{F}_T -cash additive axiom, that is

$$\forall \tilde{X} \in \tilde{\mathcal{X}}, \forall Y \in \mathcal{X}, \quad \tilde{\rho}_{\mathcal{F}_T}(\tilde{X} + Y) = \tilde{\rho}_{\mathcal{F}_T}(\tilde{X}) - Y.$$

2) $\tilde{\rho}_{\mathcal{F}_T}$ is *regular* if for any $F_T \in \mathcal{F}_T$, $\tilde{X}_T \in \tilde{\mathcal{X}}$, $\tilde{\rho}_{\mathcal{F}_T}(1_{F_T} \tilde{X}_T) = 1_{F_T} \tilde{\rho}_{\mathcal{F}_T}(\tilde{X}_T)$.

3) A cash additive risk measure $\check{\rho}$ on $\tilde{\mathcal{X}}$ is *generated from ρ via a conditional risk measure* if there exists a cash additive conditional risk measure on \mathcal{F}_T , $\tilde{\rho}_{\mathcal{F}_T}$ such that, $\check{\rho}(X_T^1, X_T^0) = \rho(-\tilde{\rho}_{\mathcal{F}_T}((X_T^1, X_T^0)))$.

It easy to see that any conditional risk measure on \mathcal{F}_T is completely determined by its value on the set $\{\theta = 1\}$. This observation leads to the following proposition.

Proposition 5.8 Any \mathcal{F}_T -measurable random function V defines a cash additive conditional risk measure on \mathcal{F}_T , $\tilde{\rho}_{\mathcal{F}_T}^V : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$, given by

$$(5.9) \quad \tilde{\rho}_{\mathcal{F}_T}^V(X_T 1_{\theta=1}) := V(X_T) \quad \text{or equivalently by} \quad \tilde{\rho}_{\mathcal{F}_T}^V((X_T^1, X_T^0)) := V(X_T^1 - X_T^0) - X_T^0.$$

Conversely, any regular and continuous from above cash additive conditional risk measure on \mathcal{F}_T , $\tilde{\rho}_{\mathcal{F}_T} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$, generates a convex random function $\tilde{V}^{\mathcal{F}_T}(\lambda) := \tilde{\rho}_{\mathcal{F}_T}(\lambda 1_{\theta=1})$ which satisfies (5.9).

Proof. Decreasing monotonicity: We refer the reader to the proof of decreasing monotonicity in Proposition 5.6. \mathcal{F}_T -cash invariance and convexity follow respectively from the definition of

⁵For conditional risk measures see Bion-Nadal (2004), Detlefsen and Scandolo (2005) and references therein.

$\tilde{\rho}_{\mathcal{F}_T}^V$ and the convexity of V .

Conversely: Define $\tilde{V}^{\mathcal{F}_T}(\omega, \lambda) := \tilde{\rho}_{\mathcal{F}_T}(\lambda 1_{\theta=1}(\omega))$. $V(\omega, \lambda)$ is \mathcal{F}_T -measurable convex and monotone decreasing functional such that $\tilde{V}^{\mathcal{F}_T}(0) = 0$ and $\tilde{V}^{\mathcal{F}_T} \in [-1, 0]$. For the regularity of $\tilde{\rho}_{\mathcal{F}_T}$ the previous definition can be extended to all the simple \mathcal{F}_T -random variables $\sum \lambda_i 1_{A_i}$, where the sets $A_i \in \mathcal{F}_T$ and $\{A_i\}_{i=1, \dots, n}$ form a partition of Ω . Hence $\tilde{\rho}_{\mathcal{F}_T}(\sum \lambda_i 1_{A_i}) = \sum 1_{A_i} \tilde{V}^{\mathcal{F}_T}(\lambda_i)$. The continuity from above of $\tilde{\rho}_{\mathcal{F}_T}$ allows to extend the definition to positive $X_T \in \mathcal{X}$ and then to any arbitrary $X_T \in \mathcal{X}$ using standard analysis tools. \square

The following theorem states the main result of this section showing that any cash sub-additive risk measure of the form $\mathcal{R}^{\rho, V} = \rho(-V)$ can be extended into a cash additive risk measure which is generated from ρ via a conditional risk measure. Conversely, any cash additive risk measure $\check{\rho}$ on $\tilde{\mathcal{X}}$ generated from ρ via a conditional risk measure is associated to a cash sub-additive risk measure of type $\mathcal{R}^{\rho, \tilde{V}^{\mathcal{F}_T}}$.

Theorem 5.9 *The cash additive risk measure $\check{\rho}^V$ in equation (5.8) is generated from ρ via the conditional risk measure $\tilde{\rho}_{\mathcal{F}_T}^V$ in (5.9) associated with V , that is*

$$(5.10) \quad \check{\rho}^V(X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0}) = \rho(-V(X_T^1 - X_T^0) + X_0) = \rho(-\tilde{\rho}_{\mathcal{F}_T}^V(X_T^1 1_{\theta=1} + X_T^0 1_{\theta=0})).$$

Moreover,

$$(5.11) \quad \mathcal{R}^{\rho, V}(X_T) = \check{\rho}^V(X_T 1_{\theta=1}) = \rho(-\tilde{\rho}_{\mathcal{F}_T}^V(X_T 1_{\theta=1})).$$

Conversely, to any cash additive risk measure $\check{\rho}(\cdot) = \rho(-\tilde{\rho}_{\mathcal{F}_T}(\cdot))$ on $\tilde{\mathcal{X}}$ generated by a cash additive conditional risk measure $\tilde{\rho}_{\mathcal{F}_T}$ on \mathcal{F}_T is associated a cash sub-additive risk measure of the following form $\mathcal{R}^{\rho, \tilde{V}^{\mathcal{F}_T}}(X_T) = \rho(-\tilde{V}^{\mathcal{F}_T}(X_T))$ where $\tilde{V}^{\mathcal{F}_T}(X_T) = \tilde{\rho}_{\mathcal{F}_T}(X_T 1_{\theta=1})$.

Proof. The proof follows easily from the previous considerations. \square

Equation (5.11) suggests that the risk of the future position X_T depends on the risk/ambiguity on the underlying asset model (the unconditional risk measure ρ) and on the risk/ambiguity on interest rates (the conditional risk measure $\tilde{\rho}_{\mathcal{F}_T}$) or more in general on the risk affecting the numéraire.

6 Optimal derivative design and inf-convolution

The problem of designing the optimal transaction between two economic agents has been largely investigated both in the insurance and in the financial literature. The risk transfer between the agents takes place through the exchange of a derivative contract and the optimal transaction is determined by a choice criterion. For example, in Barrieu and El Karoui (2006) the choice criterion is given by the minimization of the risk of the agent's exposure and the risk is assessed using forward cash additive risk measures. Using cash sub-additive risk measures we study this problem in a general framework that allows for ambiguous discount rates. We focus on the problem of the risk transfer between two agents who determine today the reserve to hedge the future exposure when the discount factor for the maturity of interest is ambiguous. To account for this ambiguity the agents collect the reserve using cash sub-additive risk measures and the decision criterium is the minimization of their reserves.

6.1 Transaction feasibility and optimization program

Let A and B be the two agents and suppose that they are evolving in a uncertain universe modeled by the probability space (Ω, \mathcal{F}_T) . Agent A is exposed towards a non-tradable risk that will impact her wealth $X_T^A \in \mathcal{X}$ at the future date T . To reduce her risk exposure and the reserve associated, A aims at issuing a derivative contract $H_T \in \mathcal{X}$ with maturity T and selling it to the agent B for a price π_0 . Agent B will enter the transaction only if this transaction reduces or leaves unchanged the reserve that she has to put aside to hedge her future exposure $X_T^B \in \mathcal{X}$. The objective is to find the optimal structure (H_T, π_0) according to the decision criterion of the agents given by their cash sub-additive risk measure \mathcal{R}_A and \mathcal{R}_B .

If the agents agree on the transaction, at time zero B pays π_0 to A . At time T the terminal wealths of the agents A and B are $X_T^A - H_T$ and $X_T^B + H_T$, respectively. A aims at minimizing the current reserve $\mathcal{R}_A(X_T^A - H_T)$ for the future exposure $X_T^A - H_T$, knowing that today she receives π_0 from B ,

$$(6.1) \quad \inf_{H_T \in \mathcal{X}, \pi_0} \mathcal{R}_A(X_T^A - H_T) - \pi_0.$$

The constraint to the optimization program (6.1) is that B enters the transaction. This happens when buying H_T for π_0 reduces or leaves unchanged the reserve $\mathcal{R}_B(X_T^B)$ that B would collect

not entering the transaction,

$$(6.2) \quad \mathcal{R}_B(X_T^B + H_T) + \pi_0 \leq \mathcal{R}_B(X_T^B).$$

The pricing rule of the H_T -structure is fully determined by the buyer B simply binding the constraint at the optimum in equation (6.2),

$$\pi_0^* = \pi_0^*(H_T) = \mathcal{R}_B(X_T^B) - \mathcal{R}_B(X_T^B + H_T).$$

This price π_0^* corresponds to an “indifference” pricing rule from the point of view of the agent B as π_0^* gives the maximum amount that agent B is ready to pay to enter the transaction. Given π_0^* , the optimization program in (6.1) becomes

$$(6.3) \quad \mathcal{R}_{A,B}(X_T^A, X_T^B) := \inf_{H_T \in \mathcal{X}} \mathcal{R}_A(X_T^A - H_T) + \mathcal{R}_B(X_T^B + H_T),$$

where the optimal transaction H_T^* attains the infimum.

6.2 Optimal transaction and inf-convolution

The risk transfer problem in equation (6.3) can be rewritten as an inf-convolution of cash sub-additive risk measures on \mathcal{X} . Indeed defining $F_T := X_T^B + H_T \in \mathcal{X}$ we have

$$(6.4) \quad \mathcal{R}_{A,B}(X_T^A, X_T^B) = \inf_{F_T \in \mathcal{X}} \{ \mathcal{R}_A(X_T^A + X_T^B - F_T) + \mathcal{R}_B(F_T) \} =: \mathcal{R}_A \square \mathcal{R}_B(X_T^A + X_T^B),$$

where \square denotes the inf-convolution. The value of $\mathcal{R}_{A,B}(X_T^A, X_T^B)$ can be interpreted as the residual measure of risk after the transaction F_T has occurred. This residual measure of risk depends on the initial exposures X_T^A and X_T^B . The transaction induces an optimal redistribution of the risks of the agents. In the following we show that $\mathcal{R}_A \square \mathcal{R}_B$ is a cash sub-additive risk measure completely characterized by \mathcal{R}^A and \mathcal{R}^B and we provide its dual representation. Also in this case, instead of using convex analysis tools to prove these results we exploit the one to one correspondence between \mathcal{R} and the cash additive risk measure $\hat{\rho}(\hat{X}_T) = \mathcal{R}(X_T - x1_T) - x$ defined on $\hat{\mathcal{X}}$ and given in equation (4.2). We show that the inf-convolution of cash sub-additive risk measures on \mathcal{X} is equal to the inf-convolution of their corresponding cash additive risk measures $\hat{\rho}$ on $\hat{\mathcal{X}}$.

Lemma 6.1 *The inf-convolution of \mathcal{R}_A and \mathcal{R}_B on \mathcal{X} in equation (6.4) corresponds to the inf-convolution of the cash additive extensions of \mathcal{R}_A and \mathcal{R}_B on $\hat{\mathcal{X}}$,*

$$(6.5) \quad \mathcal{R}_A \square \mathcal{R}_B(X_T^A + X_T^B) = \hat{\rho}_A \square \hat{\rho}_B(\hat{X}_T^A + \hat{X}_T^B), \text{ where } \hat{X}_T^A := X_T^A 1_{\theta=1}, \hat{X}_T^B := X_T^B 1_{\theta=1}.$$

$\mathcal{R}_A \square \mathcal{R}_B(X_T^A + X_T^B)$ is the infimum on $F_T \in \mathcal{X}$, while $\hat{\rho}_A \square \hat{\rho}_B(\hat{X}_T^A + \hat{X}_T^B)$ is the infimum on the pairs $(F_T, x) \in \hat{\mathcal{X}}$.

Proof. The result follows observing that any $F_T \in \mathcal{X}$ can be rewritten as $F_T = G_T - x1_T$, for some $G_T \in \mathcal{X}$ and $x \in \mathbb{R}$, and

$$\begin{aligned} \mathcal{R}_A \square \mathcal{R}_B(X_T^A + X_T^B) &= \inf_{F_T \in \mathcal{X}} \{ \mathcal{R}_A(X_T^A + X_T^B - F_T) + \mathcal{R}_B(F_T) \} \\ &= \inf_{(G_T, x) \in \mathcal{X} \times \mathbb{R}} \{ \mathcal{R}_A(X_T^A + X_T^B - (G_T - x1_T)) + \mathcal{R}_B(G_T - x1_T) \} \\ &= \inf_{\hat{G}_T = (G_T, x) \in \hat{\mathcal{X}}} \{ \hat{\rho}_A((X_T^A + X_T^B)1_{\theta=1} - \hat{G}_T) + \hat{\rho}_B(\hat{G}_T) \} = \hat{\rho}_A \square \hat{\rho}_B(\hat{X}_T^A + \hat{X}_T^B). \quad \square \end{aligned}$$

Barrieu and El Karoui (2006, Theorem 3.3) show that the inf-convolution of cash additive risk measures is a cash additive risk measure. We apply this result to $\hat{\rho}_A \square \hat{\rho}_B$. When $\hat{\rho}_A \square \hat{\rho}_B(0) > -\infty$, the inf-convolution $\hat{X} \in \hat{\mathcal{X}} \mapsto \hat{\rho}_A \square \hat{\rho}_B(\hat{X})$ is a cash additive risk measure⁶, continuous from below if one of the two risk measures is continuous from below, and with penalty function the sum of the penalties of $\hat{\rho}_A$ and $\hat{\rho}_B$. We showed that any $\hat{\rho}$ constrained to the event $\theta = 1$ defines a cash sub-additive risk measure with the same penalty function (Proposition 4.1). Then $\mathcal{R}_A \square \mathcal{R}_B$ in equation (6.5) is a cash sub-additive risk measure. We collect all the previous results in the following theorem.

Theorem 6.2 *Let \mathcal{R}_A and \mathcal{R}_B be two cash sub-additive risk measures with penalty functions α_A and α_B , respectively. Let $\mathcal{R}_{A,B}$ be the inf-convolution of \mathcal{R}_A and \mathcal{R}_B*

$$(6.6) \quad \Psi \rightarrow \mathcal{R}_{A,B}(\Psi) := \mathcal{R}_A \square \mathcal{R}_B(\Psi) = \inf_{H \in \mathcal{X}} \{ \mathcal{R}_A(\Psi - H) + \mathcal{R}_B(H) \}$$

and assume that $\mathcal{R}_{A,B}(0) > -\infty$. Then

- 1) $\mathcal{R}_{A,B}$ is a cash sub-additive risk measure which is finite for all $\Psi \in \mathcal{X}$.
- 2) The associated penalty function is given by $\forall \mu \in \mathcal{M}_{s,f}(\mathcal{F}_T), \quad \alpha_{A,B}(\mu) = \alpha_A(\mu) + \alpha_B(\mu)$.
- 3) $\mathcal{R}_{A,B}$ is continuous from below when this property holds for \mathcal{R}_A and/or \mathcal{R}_B .
- 4) The optimal derivative contract is $H^* = F^* - X_T^B$, where F^* attains the infimum in (6.4).

⁶For the interpretation of the condition $\mathcal{R}_A \square \mathcal{R}_B(0) > -\infty$ see Theorem 3.3 in Barrieu and El Karoui (2006).

7 Dynamic infinitesimal cash sub-additive risk measures

The cash sub-additive risk measures considered so far are static measures assessing the risk of the future position X_T at a given time t . In this section, we give an example of dynamic cash sub-additive risk measure on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \in [0, T]}$ is the augmented filtration associated to the d -dimensional Brownian motion $W = (W_t)_{t \in [0, T]}$. At any time $t \in [0, T]$, the risk measure assesses the riskiness of the future position X_T taking into account the information available, \mathcal{F}_t . In particular, following Peng (2004), El Karoui, Peng, and Quenez (1997), Barrieu and El Karoui (2006) and Rosazza Gianin (2006) who link backward stochastic differential equations (BSDEs) and risk measures, we show that BSDEs with suitable coefficients are cash sub-additive risk measures. The main difference with cash additive risk measures generated by BSDEs is that cash sub-additive risk measures are now recursive risk measures, that is the generator can locally depend on the level of the cash sub-additive risk measure. When the dual representation exists, the penalty function of dynamic cash sub-additive risk measures generalizes the penalty function of the static cash sub-additive risk measures in Section 3.2.

Dynamic risk measures not based on BSDEs have been recently studied by several authors such as Cvitanic and Karatzas (1999), Wang (1999), Artzner, Delbaen, Eber, Heath, and Ku (2004) Cheridito, Delbaen, and Kupper (2004), Frittelli and Rosazza Gianin (2004), Riedel (2004), Frittelli and Scandolo (2006), Cheridito, Delbaen, and Kupper (2006), Weber (2006) and Kloeppel and Schweizer (2006). Here we consider cash sub-additive risk measures generated by BSDEs.

7.1 Some results on BSDEs

Let $X_T \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ and $g(t, y, z)$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable coefficient, where \mathcal{P} is the σ -algebra of real-valued progressively measurable events. Consider the pair of squared-integrable progressively measurable processes $(Y, Z) := (Y_t, Z_t)_{t \in [0, T]}$ solution of the following BSDE associated to (g, X_T) ,

$$-dY_t = g(t, Y_t, Z_t)dt - \langle Z_t, dW_t \rangle, \quad Y_T = X_T.$$

The existence and the uniqueness of the solution $(Y_t, Z_t)_{t \in [0, T]}$ depend on the properties of the coefficient g . Pardoux and Peng (1990) prove that the solution exists and is unique when g is

uniformly Lipschitz continuous with respect to (y, z) . In this case g is called standard coefficient. When, for any given $t \in [0, T]$, g is continuous with respect to (y, z) \mathbb{P} -a.s. and $|g(t, y, z)| \leq C(1 + z^2 + y)$, $\forall (t, y, z)$ \mathbb{P} -a.s., (g with linear-quadratic growth, in the sequel), Kobylanski (2000) and Lepeltier and San Martin (1998) show that the BSDE associated with (g, X_T) has a maximal and minimal solution. Uniqueness holds under some additional assumptions.

The following theorem, called *Comparison Theorem*, is a crucial tool in the study of one-dimensional BSDEs and corresponding dynamic measures of risk.

Theorem 7.1 *Let X_T^1 and $X_T^2 \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ and g^1 and g^2 both standard (or both with linear-quadratic growth) coefficients. Let (Y^1, Z^1) and (Y^2, Z^2) be the (maximal) solutions associated to (g^1, X_T^1) and (g^2, X_T^2) , respectively. If $X_T^1 \geq X_T^2$, \mathbb{P} -a.s., and $g^1(t, Y_t^2, Z_t^2) \geq g^2(t, Y_t^2, Z_t^2)$ $d\mathbb{P} \times dt$ -a.s., then $Y_t^1 \geq Y_t^2$ a.s. $\forall t \in [0, T]$. In particular, the maximal solution is still monotone decreasing with respect to the terminal condition.*

The comparison theorem and the existence of the maximal solution ensure that, if the coefficient g is convex, the solution Y_t of the BSDE $(g, -X_T)$ is also convex when Y_t is considered as a functional of its terminal condition $-X_T$. Moreover, the existence of the maximal solution ensures the *time consistency* of $(Y_t)_{[0, T]}$, that is: $\forall 0 \leq t_1 \leq t_2 \leq T$, $Y_{t_1}(X_T) = Y_{t_1}(-Y_{t_2}(X_T))$. For the derivations of this result see, for instance, El Karoui, Peng, and Quenez (1997), Peng (2004), Barrieu and El Karoui (2006) and Rosazza Gianin (2006).

7.2 BSDEs and cash sub-additive risk measures

The link between measures of risk and BSDEs is particularly interesting because it enhances interpretation and tractability of risk measures. Barrieu and El Karoui (2006) point out that the coefficient g of BSDEs can be interpreted as infinitesimal risk measure over a time interval $[t, t + dt]$ as $\mathbb{E}_{\mathbb{P}}[dY_t | \mathcal{F}_t] = -g(t, Y_t, Z_t)dt$ where Z_t is the local volatility of the conditional risk measure, $\mathbb{V}(dY_t | \mathcal{F}_t) = |Z_t|^2 dt$. Choosing carefully the coefficient g enables to generate g -conditional risk measures that are locally compatible with the different agent beliefs.

Example 7.2 *Ambiguous interest rates.* Assume that locally $\mathbb{E}_{\mathbb{P}}[-dY_t | \mathcal{F}_t]$ is driven by the worst case scenario generated by an ambiguous discount rate $\beta = (\beta_t)_{t \in [0, T]}$, where β is an adapted

process ranging between two adapted and bounded processes $(r_t)_{t \in [0, T]}$ and $(R_t)_{t \in [0, T]}$, that is

$$\mathbb{E}_{\mathbb{P}}[-dY_t^{r, R} | \mathcal{F}_t] = \sup_{0 \leq r_t \leq \beta_t \leq R_t} (-\beta_t Y_t^{r, R}) dt.$$

$Y^{r, R}$ is the first component solution of the BSDE

$$-dY_t = -(r_t Y_t^+ - R_t Y_t^-) dt - \langle Z_t, dW_t \rangle, \quad Y_T = -X_T,$$

where $y^+ = \sup(y, 0)$ and $y^- = \sup(-y, 0)$. More precisely, since $(r_t)_{t \in [0, T]}$ and $(R_t)_{t \in [0, T]}$ are assumed to be bounded, $(Y^{r, R}, Z^{r, R})$ is the unique solution of the standard BSDE with convex Lipschitz coefficient

$$(7.1) \quad g^{r, R}(t, y) = R_t y^- - r_t y^+ = \sup_{r_t \leq \beta_t \leq R_t} (-\beta_t y).$$

Notice that $y \mapsto g^{r, R}(t, y)$ is a monotone non increasing function. To provide the intuition on this risk measure, we apply the comparison theorem to the coefficients $g^{r, R}(t, y)$ and $g(t, y) = (-\beta_t y)$, $\beta_t \in [r_t, R_t]$, with the same terminal condition $-X_T$. Since $g^{r, R}(t, y) \geq (-\beta_t y)$, $Y_t^{r, R} \geq Y_t^\beta$ where Y^β is the solution of the linear BSDE

$$-dY_t = -\beta_t Y_t dt - \langle Z_t, dW_t \rangle, \quad Y_T = -X_T,$$

and it can be represented as $Y_t^\beta = \mathbb{E}_{\mathbb{P}}[e^{-\int_t^T \beta_s ds} (-X_T) | \mathcal{F}_t]$, $\forall t \in [0, T]$. Then it follows that $Y_t^{r, R} \geq \text{ess sup}_{0 \leq r_t \leq \beta_t \leq R_t} Y_t^\beta$. As the process $\bar{\beta}_t = R_t 1_{Y_t^{r, R} \leq 0} + r_t 1_{Y_t^{r, R} > 0}$ achieves the maximum of $\sup_{r_t \leq \beta_t \leq R_t} (-\beta_t Y_t^{r, R}) = -\bar{\beta}_t Y_t^{r, R}$, then the equality $Y_t^{r, R} = Y_t^{\bar{\beta}_t}$ holds. Thus, the dual representation of $Y_t^{r, R}$ follows

$$Y_t^{r, R} = Y_t^{\bar{\beta}_t} = \text{ess sup}_{0 \leq r_t \leq \beta_t \leq R_t} \mathbb{E}_{\mathbb{P}}[e^{-\int_t^T \beta_s ds} (-X_T) | \mathcal{F}_t].$$

Notice that, for any $t \in [0, T]$, $Y_t^{r, R}$ is dominated, but in general not equal to the conditional risk measure $\mathcal{R}_t^{D^R, D^r}$ associated with the worst case discounted factors $D_{t, T}^R \leq D_{t, T} \leq D_{t, T}^r$, where $D_{t, T}^R = \exp\{-\int_t^T R_s ds\}$ and $D_{t, T}^r = \exp\{-\int_t^T r_s ds\}$,

$$(7.2) \quad Y_t^{r, R}(-X_T) \leq \mathcal{R}_t^{D^R, D^r}(X_T) = \mathbb{E}_{\mathbb{P}}[D_{t, T}^R (-X_T)^- + D_{t, T}^r (-X_T)^+ | \mathcal{F}_t].$$

$\mathcal{R}^{D^R, D^r} := (\mathcal{R}_t^{D^R, D^r})_{t \in [0, T]}$ is a cash sub-additive risk measure which is not time consistent in contrast to $Y^{r, R} = (Y_t^{r, R})_{t \in [0, T]}$.

In the sequel we consider risk measures generated by BSDEs which generalize Example 7.2. For the remain part of the paper $g(t, y, z)$ denotes a convex generator in (y, z) , standard or with linear growth with respect to y and quadratic growth in z . The comparison theorem ensures that the (maximal) solution (Y, Z) associated with a $(g, -X_T)$ exists and, for any $t \in [0, t]$, Y_t is convex and decreasing with respect to the final condition $-X_T$.

The coefficient $g^{r, R}(t, y)$ in equation (7.1) depends on y in a convex decreasing way. As observed by Peng (2004) and Barrieu and El Karoui (2006), this is never the case for conditional cash additive risk measures generated by BSDEs. Under some mild additional assumptions, Peng (2004) shows that, for any $t \in [0, T]$, the (maximal) solution Y_t associated with $(g, -X_T)$ is cash additive as functional of its terminal condition if and only if g does not depend on y for any $t \in [0, T]$. Barrieu and El Karoui (2006) study these cash additive solutions as a dynamic risk measure $(\rho_t(X_T))_{t \in [0, T]}$, $\rho_t(X_T) = Y_t(-X_T)$, that they call g -conditional risk measures⁷.

In the following proposition we show that conditional risk measures generated by BSDEs are cash sub-additive when the convex coefficient $g(t, y, z)$ depends on both y and z and is decreasing with respect to y .

Proposition 7.3 *If the convex $g(t, y, z)$ is decreasing with respect to y then the (maximal) solution Y_t of the BSDE associated with $(g, -X_T)$ is a conditional cash sub-additive risk measure, $\mathcal{R}_t^g(X_T) = Y_t$ and $\mathcal{R}^g = (\mathcal{R}_t^g(X_T))_{t \in [0, T]}$ is a time consistent cash sub-additive risk measure. We call $\mathcal{R}^g = (\mathcal{R}_t^g(X_T))_{t \in [0, T]}$ g -conditional cash sub-additive risk measure.*

Proof. For the convexity and the decreasing monotonicity of Y_t with respect to the terminal condition see, for instance, El Karoui and Quenez (1996) and Peng (1997).

Cash sub-additivity: Consider the BSDE satisfied by $\mathcal{R}_t^g(X_T + m1_T) + m = Y_t^m$,

$$-dY_t^m = g^m(t, Y_t^m, Z_t^m)dt - \langle Z_t^m, dW_t \rangle, \quad Y_T^m = -X_T.$$

Since $g^m(t, y, z) = g(t, y - m, z)$, then $g^m(t, y, z)$ is increasing in m (as g is decreasing in y). From the comparison theorem it follows that $\mathcal{R}_t^g(X_T + m1_T) + m = Y_t^m$ is increasing in m . \square

⁷If $g(t, 0) = 0$ for any $t \in [0, T]$, the g -conditional risk measures coincide with the non linear expectation originally studied by Peng (2004); see also Rosazza Gianin (2006).

7.3 Dual Representation

In this section we focus on a dual representation for g -conditional cash sub-additive risk measures \mathcal{R}^g as in the static case. For the cash additive g -conditional risk measures such a representation has been derived in Barrieu and El Karoui (2006). The next result is a straightforward generalization of their results.

The key tool to obtain dual representations is the Legendre transform of the generator g defined by

$$G(t, \beta, \mu) := \sup_{(y, z) \in \mathbb{R} \times \mathbb{R}^d} \{-\beta y - \langle \mu, z \rangle - g(t, y, z)\}.$$

The following lemma summarizes the properties of G and g .

Lemma 7.4 *Let g be a continuous convex function on $\mathbb{R} \times \mathbb{R}^d$ satisfying the growth control: there exist two positive constants $C > 0$ and $k > 0$ such that $|g(t, y, z)| \leq |g(t, 0, 0)| + C|y| + \frac{k}{2}|z|^2$.*

i) Then the Legendre transform of g , $G(t, \beta, \mu)$, takes infinite values if $\beta \notin [0, C]$. Moreover,

$$(7.3) \quad G(t, \beta, \mu) \geq -|g(t, 0, 0)| + \frac{1}{2k}|\mu|^2.$$

ii) Since g is continuous, for any $t \in [0, T]$, $g(t, Y_t, Z_t) = \sup_{\beta, \mu} \{-\beta Y_t - \langle \mu_t, Z_t \rangle - G(t, \beta_t, \mu_t)\}$.

The maximum is achieved at $(\bar{\beta}_t, \bar{\mu}_t)$ with $0 \leq \bar{\beta}_t \leq C$ and $|\bar{\mu}_t|^2 \leq A(|g(t, 0, 0)| + C|Y_t|) + B|Z_t|^2$, for some A and B positive constants.

Proof. *i)* $G(t, \beta, \mu) \geq -\beta y - g(t, y, 0) \geq -\beta y - |g(t, 0, 0)| - C|y|$. Then, if $|\beta| > C$, $\sup_{y \in \mathbb{R}} \{-\beta y - C|y|\} = +\infty$. Moreover, since $g(t, y, z)$ is monotone decreasing with respect to y , $-g(t, y, 0) \geq -g(t, 0, 0)$, $\forall y > 0$ and $G(t, \beta, \mu) \geq -\beta y - g(t, 0, 0)$, $\forall y > 0$. Then $G(t, \beta, \mu) = +\infty$ if $\beta < 0$. To prove the inequality (7.3), we observe that $G(t, \beta, \mu) \geq \langle \mu, -z \rangle - g(t, 0, z) \geq \langle \mu, -z \rangle - |g(t, 0, 0)| - \frac{k}{2}|z|^2$. As $\max_{z \in \mathbb{R}^d} \{\langle \mu, -z \rangle - \frac{k}{2}|z|^2\} = \frac{1}{2k}|\mu|^2$ the result follows.

ii) Standard results in convex analysis show that, since g is continuous, the duality between g and G holds true and the maximum is achieved.

To show the inequality in *ii)*, we choose a constant ε such that $0 < \varepsilon < \frac{1}{2k}$ and we use the

inequality (7.3),

$$\begin{aligned}
\left(\frac{1}{2k} - \varepsilon\right) |\bar{\mu}_t|^2 &\leq |g(t, 0, 0)| + G(t, \bar{\beta}_t, \bar{\mu}_t) - \varepsilon |\bar{\mu}_t|^2 \\
&\leq |g(t, 0, 0)| - \bar{\beta}_t Y_t + \langle \bar{\mu}_t, -Z_t \rangle - g(t, Y_t, Z_t) - \varepsilon |\bar{\mu}_t|^2 \\
&\leq 2|g(t, 0, 0)| + 2C|Y_t| + \frac{k}{2}|Z_t|^2 + \sup_{\mu_t} \{\langle \mu_t, -Z_t \rangle - \varepsilon |\mu_t|^2\}.
\end{aligned}$$

As $\max_{\mu_t \in \mathbb{R}} \{\langle \mu_t, -Z_t \rangle - \varepsilon |\mu_t|^2\} = \frac{|Z_t|^2}{4\varepsilon}$, then $(\frac{1}{2k} - \varepsilon) |\bar{\mu}_t|^2 \leq 2|g(t, 0, 0)| + 2C|Y_t| + (\frac{k}{2} + \frac{1}{4\varepsilon}) |Z_t|^2$, which proves the inequality. \square

Now we introduce the class of probability measures that appears in the dual representation. As in Barrieu and El Karoui (2006) the reference is the Girsanov theorem for the BMO-exponential martingales such as defined in Kazamaki (1994),

$$\Gamma_t^\mu = \mathcal{E}(M_t^\mu) = \exp \left(- \int_0^t \mu_s dW_s - \frac{1}{2} \int_0^t |\mu_s|^2 ds \right),$$

where $M_t^\mu = \int_0^t \mu_s dW_s$ is a $\text{BMO}(\mathbb{P})$ -martingale, that is μ belongs to $\text{BMO}(\mathbb{P})$,

$$\text{BMO}(\mathbb{P}) := \{\psi \in \mathcal{H}^2 \text{ such that } \exists C > 0 : \mathbb{E}_{\mathbb{P}} \left[\int_t^T \psi_s^2 ds | \mathcal{F}_t \right] \leq C \text{ a.s., } \forall t \in [0, T]\},$$

where $\mathcal{H}^2 = \{\psi \in \mathcal{P}_1 \text{ such that } \mathbb{E}[\int_0^T \psi^2 ds] < \infty\}$. Using Kazamaki (1994, Section 3.3), Γ_T^μ is the likelihood of an equivalent probability measure on \mathcal{F}_T with respect to \mathbb{P} defined by $d\mathbb{Q}^\mu = \Gamma_T^\mu d\mathbb{P}$. Moreover, if $v \in \text{BMO}(\mathbb{P})$ then $v \in \text{BMO}(\mathbb{Q}^\mu)$. Recall that Γ_t^μ is the solution of the forward stochastic differential equation

$$d\Gamma_t^\mu = \Gamma_t^\mu \langle -\mu_t, dW_t \rangle, \quad \Gamma_0^\mu = 1.$$

Now we establish the duality theorem.

Theorem 7.5 *Let g be a convex coefficient, decreasing with respect to y and with growth $|g(t, y, z)| \leq |g(t, 0, 0)| + C|y| + \frac{k}{2}|z|^2$. Moreover, assume that there exists a constant $K > 0$ such that $\mathbb{E}[\int_t^T |g(s, 0, 0)| ds | \mathcal{F}_t] \leq K, \forall t \in [0, T]$. Then the (maximal) solution (Y, Z) of the BSDE*

$$-dY_t = g(t, Y_t, Z_t) - \langle Z_t, dW_t \rangle, \quad Y_T = -X_T, \quad X_T \in L^\infty(\mathbb{P}),$$

is bounded and Z is in $\text{BMO}(\mathbb{P})$. Let $G(t, y, z)$ be the Fenchel transform of g and

$$\mathcal{A} := \{(\beta_t, \mu_t)_{t \in [0, T]} | G(t, \beta_t, \mu_t) < +\infty, 0 \leq \beta_t \leq C, \forall t \in [0, T] \text{ and } \mu \in \text{BMO}(\mathbb{P})\}.$$

Then, the g -conditional cash sub-additive risk measure $\mathcal{R}^g = (\mathcal{R}_t^g(X_T))_{t \in [0, T]}$, $\mathcal{R}_t^g(X_T) = Y_t$, has the following dual representation

$$(7.4) \quad \mathcal{R}_t^g(X_T) = \text{ess sup}_{(\beta, \mu) \in \mathcal{A}} \mathbb{E}_{\mathbb{Q}^\mu} \left[e^{-\int_t^T \beta_s ds} (-X_T) - \int_t^T e^{-\int_t^s \beta_u du} G(s, \beta_s, \mu_s) ds \middle| \mathcal{F}_t \right].$$

Remark 7.6 The dual representation of \mathcal{R}^g in equation (7.4) is similar to the dual representation of static cash sub-additive risk measures. Here, the sub-probability measures are replaced by the \mathcal{F}_t -conditional sub-probability measures $R^{\beta, \mu}$

$$\frac{dR^{\beta, \mu}}{d\mathbb{P}} \middle| \mathcal{F}_t := \exp \left(- \int_t^T \mu_s dW_s - \frac{1}{2} \int_t^T |\mu_s|^2 ds - \int_t^T \beta_s ds \right)$$

and the penalty function becomes

$$\alpha_t(R^{\beta, \mu}) := R^{\beta, \mu} \left(\int_t^T e^{-\int_t^s \beta_u du} G(s, \beta_s, \mu_s) ds \middle| \mathcal{F}_t \right).$$

Proof. *i)* To show that $Z \in \text{BMO}(\mathbb{P})$ we refer the reader to the proof in Barrieu and El Karoui (2006).

ii) From the Girsanov theorem for BMO-martingales we know that for any $0 \leq \beta_t \leq C$, $\mu \in \text{BMO}(\mathbb{P})$, $dW_t^\mu = dW_t + \mu_t dt$ is a \mathbb{Q}^μ -Brownian motion and

$$\begin{aligned} -dY_t &= g(t, Y_t, Z_t) - \langle Z_t, dW_t \rangle \\ &= [g(t, Y_t, Z_t) + \beta_t Y_t + \langle \mu_t, Z_t \rangle] dt - \beta_t Y_t dt - \langle Z_t, dW_t^\mu \rangle. \end{aligned}$$

Then it follows

$$\begin{aligned} Y_t(-X_T) &= \mathbb{E}_{\mathbb{Q}^\mu} \left[e^{-\int_t^T \beta_s ds} (-X_T) + \int_t^T e^{-\int_t^s \beta_u du} [g(s, Y_s, Z_s) + \beta_s Y_s + \langle \mu_s, Z_s \rangle] ds \middle| \mathcal{F}_t \right] \\ (7.5) \quad &\geq \mathbb{E}_{\mathbb{Q}^\mu} \left[e^{-\int_t^T \beta_s ds} (-X_T) - \int_t^T e^{-\int_t^s \beta_u du} G(s, \beta_s, \mu_s) ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

To prove the last equality in (7.5) at the optimal control $(\bar{\beta}, \bar{\mu})$,

$$G(t, \bar{\beta}, \bar{\mu}) = -\bar{\beta}_t Y_t - \langle \bar{\mu}_t, Z_t \rangle - g(t, Y_t, Z_t), \quad \forall t \in [0, T],$$

we need to verify that $(\bar{\beta}, \bar{\mu})$ is admissible. Since $0 \leq \bar{\beta}_t \leq C$, we only need to verify that $\bar{\mu}$ is in $\text{BMO}(\mathbb{P})$. We use the inequality in Lemma 7.4, $|\bar{\mu}_t|^2 \leq A(|g(t, 0, 0)| + c|Y_t|) + B|Z_t|^2$. Since $|g(t, 0, 0)|^{1/2}$ belongs to $\text{BMO}(\mathbb{P})$, Y is bounded and $Z \in \text{BMO}(\mathbb{P})$, then $\bar{\mu} \in \text{BMO}(\mathbb{P})$,

$$Y_t(-X_T) = \mathcal{R}^g(X_T) = \mathbb{E}_{\mathbb{Q}^{\bar{\mu}}} \left[e^{-\int_t^T \bar{\beta}_s ds} (-X_T) - \int_t^T e^{-\int_t^s \bar{\beta}_u du} G(s, \bar{\beta}_s, \bar{\mu}_s) ds \middle| \mathcal{F}_t \right]$$

and this establishes the dual representation. \square

8 Conclusion

We propose a new class of risk measures called cash sub-additive risk measures which accounts for the risk/ambiguity on interest rates when assessing the risk of future financial, nonfinancial and insurance positions. This goal is achieved by relaxing the debated cash additive axiom into the cash sub-additive axiom. We provide several examples of the new risk measures in the static and the dynamic frameworks, such as the put options premium and the robust expected utilities. In the dynamic framework cash sub-additive risk measures are generated by BSDEs enhancing their tractability and interpretability. Cash sub-additive risk measures represent a promising research area as these risk measures overcome the issues arising from the cash additive axiom.

References

- Artzner, P., F. Delbaen, J. Eber, and D. Heath, 1997, “Thinking coherently,” *Risk*, 10, 68–71.
- , 1999, “Coherent measures of risk,” *Mathematical Finance*, 9, 203–228.
- Artzner, P., F. Delbaen, J. Eber, D. Heath, and H. Ku, 2004, “Coherent multiperiod risk adjusted values and Bellman’s principle,” Preprint, ETH, Zurich.
- Barrieu, P., and N. El Karoui, 2006, “Pricing, hedging and optimally designing derivatives via minimization of risk measures,” in *Volume on Indifference Pricing*, ed. by R. Carmona. Princeton University Press, to appear.
- Bion-Nadal, J., 2004, “Conditional risk measure and robust representation of convex conditional risk measures,” CMAP Preprint, Ecole Polytechnique.
- Carr, P., H. Geman, and D. B. Madan, 2001, “Pricing and hedging in incomplete markets,” *Journal of Financial Economics*, 62, 131–167.
- Cheridito, P., F. Delbaen, and M. Kupper, 2004, “Coherent and convex monetary risk measures for bounded cdlg processes,” *Stochastic Processes and their Applications*, 112, 1–22.
- , 2006, “Dynamic monetary risk measures for bounded discrete-time processes,” *Electronic Journal of Probability*, 11, 57–106.
- Cheridito, P., and M. Kupper, 2006, “Composition of time-consistent dynamic monetary risk measures in discrete time,” preprint, Princeton University.
- Cvitanic, J., and I. Karatzas, 1999, “On dynamic risk measures,” *Finance and Stochastics*, 3, 451–482.
- Delbaen, F., 2001, “Coherent risk measures,” Lectures notes, Scuola Normale Superiore di Pisa.
- , 2002, “Coherent measures of risk on general probability spaces,” in *Advances in finance and stochastics*, ed. by K. Sandmann, and P. Schönbucher. New York: Springer-Verlag.
- Detlefsen, K., and G. Scandolo, 2005, “Conditional and dynamic convex risk measures,” *Finance and Stochastics*, 9, 539–561.
- El Karoui, N., S. Peng, and M. C. Quenez, 1997, “Backward stochastic differential equations in finance,” *Mathematical Finance*, 7, 1–71.

- El Karoui, N., and M. C. Quenez, 1996, “Non-linear pricing theory and backward stochastic differential equations in financial mathematics,” in *Lecture Notes in Mathematics 1656*, ed. by W. Runggaldier. New York: Springer-Verlag.
- Föllmer, H., and A. Schied, 2002a, “Convex measures of risk and trading constraints,” *Finance and Stochastics*, 6, 429–447.
- , 2002b, *Stochastic finance: An introduction in discrete time*. De Gruyter Studies in Mathematics, Berlin, Germany.
- Frittelli, M., 2000, “Representing sublinear risk measures and pricing rules,” Working paper, Univesità Bicocca di Milano, Italy.
- Frittelli, M., and E. Rosazza Gianin, 2002, “Putting order in risk measures,” *Journal of Banking and Finance*, 26, 1473–1486.
- , 2004, “Dynamic convex risk measures,” in *Risk Measures for the 21st Century*, ed. by G. Szegö. New York: John Wiley and Sons.
- Frittelli, M., and G. Scandolo, 2006, “Risk measures and capital requirement for processes,” *Mathematical Finance*, 16, 589–612.
- Jarrow, R., 2002, “Put option premium and coherent risk measures,” *Mathematical Finance*, 12, 135–142.
- Kazamaki, N., 1994, “Continuous exponetial martingales and BMO,” in *Lecture Notes in Mathematics 1579*, ed. by W. Runggaldier. Berlin: Springer-Verlag.
- Kloeppel, S., and M. Schweizer, 2006, “Dynamic indifference valuation via convex risk measures,” *Mathematical Finance*, to appear.
- Kobylanski, M., 2000, “Backward stochastic differential equations and partial differential equations with quadratic growth,” *Annals of Probability*, 28, 558–602.
- Krätschmer, V., 2005, “Robust representation of convex risk meaures by probability measures,” *Finance and Stochastics*, 9, 597–608.
- , 2007, “On σ -additive robust representation of convex risk measures for unbounded financial positions in the presence of uncertainty about the market model,” Discussion paper, Berlin University of Technology.

- Lepeltier, J.-P., and J. San Martin, 1998, “Existence for BSDE with superlinear-quadratic coefficients,” *Stochastics and Stochastic Reports*, 63, 227–240.
- Maccheroni, F., M. Marinacci, and A. Rustichini, 2004, “Ambiguity aversion, robustness, and the variational representation of preferences,” *Econometrica*, to appear.
- Pardoux, E., and S. Peng, 1990, “Adapted solution of a backward stochastic differential equation,” *Systems and Control Letters*, 14, 55–61.
- Peng, S., 1997, “Backward SDE and related g -expectations,” in *Backward stochastic equations; Stochastic differential equations, Pitman Research Notes in Mathematics Series. 364*, ed. by N. El Karoui, and L. Mazliak. Harlow: Longman.
- , 2004, “Nonlinear expectations, nonlinear evaluations and risk measures,” in *Lecture Notes in Mathematics 1856*, ed. by W. Runggaldier. New York: Springer-Verlag.
- Riedel, F., 2004, “Dynamic coherent risk measures,” *Stochastic Processes and their Applications*, 112, 185–200.
- Rosazza Gianin, E., 2006, “Risk measures via g -expectations,” *Insurance: Mathematics and Economics*, 39, 19–34.
- Rouge, R., and N. El Karoui, 2000, “Pricing via utility maximization and entropy,” *Mathematical Finance*, 10, 259–276.
- Schied, A., 2004, “Optimal investments for robust utility functionals in incomplete markets models,” Working paper, University of Berlin.
- Staum, J., 2004, “Fundamental theorems of asset pricing for good deal bounds,” *Mathematical Finance*, 14, 141–161.
- Wang, T., 1999, “A class of dynamic risk measures,” Working paper, University of British Columbia.
- Weber, S., 2006, “Distribution-invariant risk measures, information, and dynamic consistency,” *Mathematical Finance*, 16, 419–441.